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## Arc Length Parameterization

*. most of the time we can evaluate these integrals by a substitution or factoring into a perfect square

1. Find the length of the following curves over the following intervals.

(a) $\vec{r}(t) = \langle 2t, \ln t, t^2 \rangle$, $1 \le t \le 4$

$\vec{r}\,'(t) = \langle 2, \frac{1}{t}, 2t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{2^2 + (\frac{1}{t})^2 + (2t)^2} = \sqrt{4 + \frac{1}{t^2} + 4t^2} = \sqrt{\frac{1}{t^2}(\underbrace{4t^4 + 4t^2 + 1}_{\text{perfect square}})}$

$= \sqrt{\frac{1}{t^2}(2t^2+1)^2} = \frac{1}{t}(2t^2+1) = 2t + \frac{1}{t}$

$L = \int_1^4 \|\vec{r}\,'(t)\|\, dt = \int_1^4 (2t + \frac{1}{t})\, dt = (t^2 + \ln t)\Big|_1^4 = 4^2 + \ln 4 - (1^2 + \cancel{\ln 1}^{\,0})$

$= 16 + \ln 4 - 1 = 15 + \ln 4$

(b) $\vec{r}(t) = \langle t, 4t^{3/2}, 2t^{3/2} \rangle$, $0 \le t \le 3$.

$\vec{r}\,'(t) = \langle 1, 4\cdot\frac{3}{2}t^{1/2}, 2\cdot\frac{3}{2}t^{1/2} \rangle = \langle 1, 6t^{1/2}, 3t^{1/2} \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1 + (6t^{1/2})^2 + (3t^{1/2})^2} = \sqrt{1 + 36t + 9t} = \sqrt{1+45t}$

$L = \int_0^3 \|\vec{r}\,'(t)\|\, dt = \int_0^3 \sqrt{1+45t}\, dt$

substitution: take $u = 1+45t$
$du = 45dt$
$\Rightarrow dt = \frac{du}{45}$

*bounds: when $t=0$, $u=1$
$t=3$, $u = 1+45(3) = 136$

$= \int_1^{136} \sqrt{u}\, \frac{du}{45} = \frac{1}{45}\int_1^{136} u^{1/2}\, du$

$= \frac{1}{45}\cdot\frac{2}{3}u^{3/2}\Big|_1^{136} = \frac{2}{135}u^{3/2}\Big|_1^{136}$

$= \frac{2}{135}\left((136)^{3/2} - 1^{3/2}\right) = \frac{2}{135}\left((136)^{3/2} - 1\right)$

2. Consider the line parallel to $\vec{v} = \langle 3,4,12\rangle$ and passing through the point $(0,1,-12)$.

   (a) Find an equation for this line of the form $\vec{r}(t) = \ldots$.

   (b) Find the velocity of a point whose motion is described by your equation above.

   (c) Find the speed and the arc length from a point $t = 0$.

   (d) Reparameterize with respect to arc length.

(a) $\vec{r}(t) = \langle 0,1,-12\rangle + t\langle 3,4,12\rangle = \langle 3t, 1+4t, -12+12t\rangle$

(b) $\vec{v}(t) = \vec{r}'(t) = \langle 3,4,12\rangle$

(c) speed $= \|\vec{v}(t)\| = \|\vec{r}'(t)\| = \|\langle 3,4,12\rangle\| = \sqrt{3^2+4^2+12^2} = \sqrt{9+16+144} = \sqrt{169} = 13$

Arc length function: $s(t) = \int_0^t \|\vec{r}'(u)\|\,du = \int_0^t 13\,du = 13u\Big|_0^t = 13t$

(d) Solve for $t$ in terms of $s$: $s = 13t \Rightarrow t = \frac{s}{13}$

Substitute back into $\vec{r}$: $\boxed{\vec{r}(s) = \left\langle \frac{3s}{13}, 1+\frac{4s}{13}, -12+\frac{12s}{13}\right\rangle}$

observe the velocity now is $\vec{v}(s) = \vec{r}'(s) = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle$

which has magnitude $\|\vec{v}(s)\| = \|\vec{r}'(s)\| = \sqrt{\frac{3^2}{13^2} + \frac{4^2}{13^2} + \frac{12^2}{13^2}} = 1$

3. Now choose two other points along the same line, and find the equation of the line using these two other points. * Answers will vary.

   (a) Find an equation for this line of the form $\vec{r}(t) = \ldots$.

   (b) Find the velocity of a point whose motion is described by your equation above.

   (c) Find the speed and the arc length from a point $t = 0$.

   (d) Reparameterize with respect to arc length.

(a) Two points: plug in $t=0$ and $t=2$ into 2(a) formula to get two other points on the same line: $(0,1,-12)$, $(6,9,12)$ has direction $\vec{u} = \langle 6-0, 9-1, 12-(-12)\rangle = \langle 6,8,24\rangle$

This line has the formula $\vec{r}(t) = \langle 0,1,-12\rangle + t\langle 6,8,24\rangle = \langle 6t, 8t+1, 24t-12\rangle$

(b) $\vec{v}(t) = \vec{r}\,'(t) = \langle 6, 8, 24\rangle$

(c) speed $= \|\vec{v}(t)\| = \|\vec{r}\,'(t)\| = \sqrt{6^2+8^2+24^2} = \sqrt{2^2 3^2 + 2^2 4^2 + 2^2 12^2} = 2\sqrt{3^2+4^2+12^2} = 26$

arc length function $s(t) = \int_0^t \|\vec{r}\,'(u)\|\,du = \int_0^t 26\,du = 26t$

(d) solve for $t$ in terms of $s$: $s = 26t \rightarrow t = \frac{s}{26}$

plug back into $\vec{r}(t)$: $\vec{r}(s) = \langle \frac{6s}{26}, \frac{8s}{26}+1, \frac{24s}{26}-12\rangle$

* notice the velocity of $\vec{r}(s)$ is $\vec{v}(s) = \langle \frac{6}{26}, \frac{8}{26}, \frac{24}{26}\rangle = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$ and has magnitude 1

What do you notice about the velocity vector in both cases? How does it relate to the original vector you started with? How can you explain the similarities and differences?

- The velocity vector of the reparametrized equations are the same and has magnitude 1 (* note: depending on how you chose your points the velocity vectors of the reparametrized equations could be going in opposite directions, BUT should still have magnitude 1)
- Velocity vectors of reparametrized equations are a scalar multiple of the original velocity vectors