WEEK 6 SOLUTIONS

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Extra explanation of $\varepsilon-\delta$ proofs...

Pre-#1) Let's look at a 2D example first, and then we will generalize to the given problem.

def. $\lim_{x\to a} f(x) = L \iff$ For every $\varepsilon>0$ there is a $\delta>0$ so that $|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$

* Note: $|x-a|<\delta \Rightarrow -\delta < x-a < \delta \Rightarrow a-\delta < x < a+\delta$

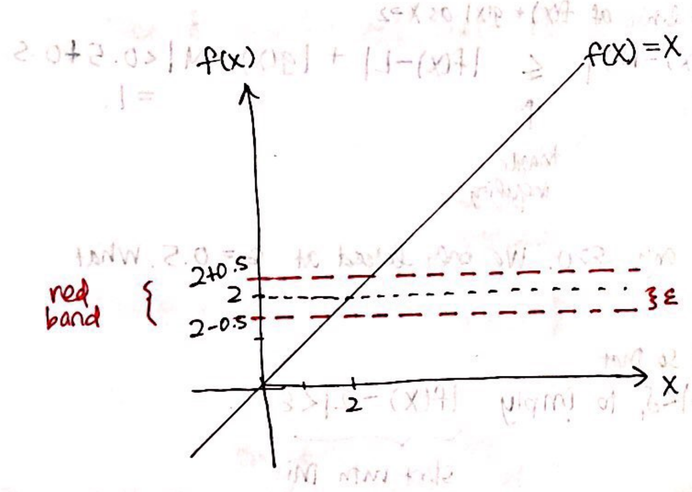
$|f(x)-L|<\varepsilon \Rightarrow -\varepsilon < f(x)-L < \varepsilon \Rightarrow L-\varepsilon < f(x) < L+\varepsilon$.

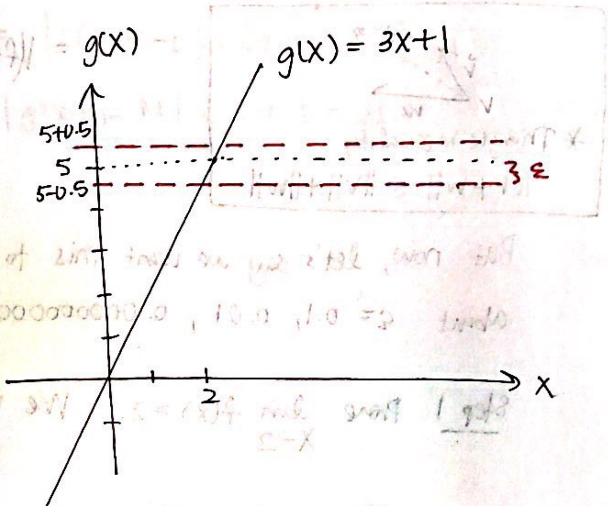
In Layman's terms: how close do I have to be to a in order to guarantee my function values are close to L?

EX $f(x)=x$, $g(x)=3x-1$, $a=2$.

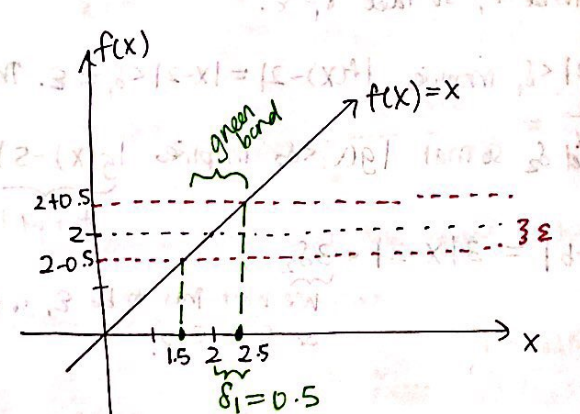
We have previous knowledge that $\lim_{x\to 2} x = 2$ and $\lim_{x\to 2} 3x-1 = 3(2)-1 = 5$.

let $\varepsilon = 0.5$.

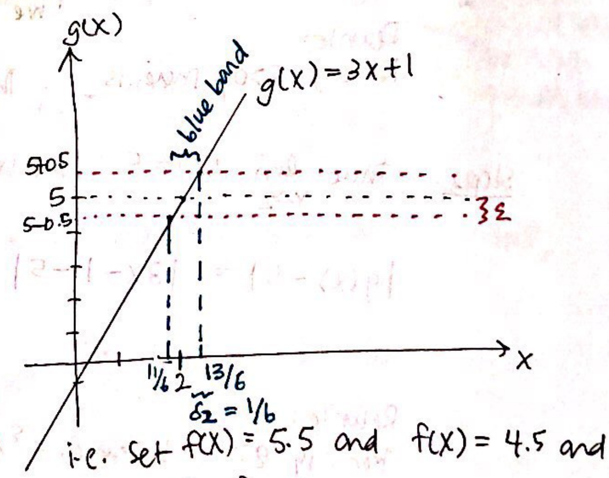




We need to find the x values so that we stay inside the red lines/"band".



i.e. set $f(x)=2.5$ and $f(x)=1.5$ and solve for x



i.e. set $f(x) = 5.5$ and $f(x) = 4.5$ and solve for x

Pre-#1 Continued

These pictures tell us that for $\varepsilon = 0.5$

- $|x-2| < \underset{\delta_1}{\underset{\uparrow}{0.5}}$ guarantees that $|f(x) - \underset{\substack{\uparrow \\ \text{this is the limit of } f(x) \text{ as } x \to 2}}{L}| = |f(x) - 2| < \overset{\varepsilon}{\overset{\downarrow}{0}}.5.$
- $|x-2| < \underset{\delta_2}{\underset{\uparrow}{\frac{1}{6}}}$ guarantees that $|g(x) - \underset{\substack{\uparrow \\ \text{limit of } g(x) \text{ as } x \to 2}}{M}| = |g(x) - 5| < 0.5$

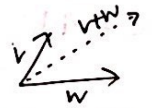
So what δ do we need to choose so that $\lim_{x \to 2} (f(x) + g(x)) = L+M = 2+5 = 7$?

We need to choose δ so that the above two statements hold, or that we stay inside both the green and blue bands in the previous pictures. This means we are going to pick the smaller of the two δ_1 and δ_2 values $\Rightarrow$ $\delta = \min\{\delta_1, \delta_2\} = \min\{0.5, \frac{1}{6}\} = \frac{1}{6}$.

Then for $\varepsilon = 0.5$,

$|x-2| < \delta = \frac{1}{6}$ implies $|f(x) + g(x) - \underbrace{(L+M)}_{\substack{\uparrow \\ \text{The limit of } f(x)+g(x) \text{ as } x\to 2}}| = |f(x) + g(x) - L - M|$

$$= |(f(x) - L) + (g(x) - M)| \underset{\substack{\uparrow \\ \text{triangle} \\ \text{inequality}}}{\leq} |f(x) - L| + |g(x) - M| < 0.5 + 0.5 = 1.$$



$*$ Triangle inequality
$\|v+w\| \leq \|v\| + \|w\|$.

But now, let's say we want this to be true for any $\varepsilon > 0$. We only looked at $\varepsilon = 0.5$. What about $\varepsilon = 0.1, 0.01, 0.0000000001$, etc.?

<u>Step 1</u> Prove $\lim_{x \to 2} f(x) = 2$. We want to find δ so that $|x-2| < \delta_1$ implies $\underbrace{|f(x) - 2| < \varepsilon}_{\text{start with this}}$.

$|f(x) - 2| = |x-2| < \delta_1$ $\leftarrow$ we want this to be ε, so take $\delta_1 = \varepsilon$.

Rewrite:
For any $\varepsilon > 0$, choose $\delta_1 = \varepsilon$. Then $|x-2| < \delta_1$ implies $|f(x) - 2| = |x-2| < \delta_1 = \varepsilon$. Thus $\lim_{x \to 2} x = 2$.

<u>Step 2.</u> Prove $\lim_{x \to 2} g(x) = 5$. We want to find δ_2 so that $|x-2| < \delta_2$ implies $\underbrace{|g(x) - 5| < \varepsilon}_{\text{start w/ this}}$.

$|g(x) - 5| = |3x - 1 - 5| = |3x - 6| = 3|x-2| < \underbrace{3\delta_2}$ we want this to be ε, so take $3\delta_2 = \varepsilon$ or $\delta_2 = \varepsilon/3$.

Rewrite:
For any $\varepsilon > 0$, choose $\delta_2 = \varepsilon/3$. Then $|x-2| < \delta_2$ implies $|g(x) - 5| = |3x - 1 - 5| = 3|x-2|$
$< 3\delta_2 = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$.

Pre-#1 continued

Step 3 Prove $\lim_{x\to 2}(f(x)+g(x)) = 7$. Take $\delta = \min\{\delta_1, \delta_2\} = \min\{\varepsilon, \frac{\varepsilon}{3}\} = \frac{\varepsilon}{3}$. Then

$$|f(x)+g(x)-(L+M)| = |(f(x)-L)+(g(x)-M)| = |(x-2)+(3x-1-5)|$$
$$\leq |x-2|+|3x-6| < \varepsilon+\varepsilon = 2\varepsilon.$$

triangle inequality

Therefore $\lim_{x\to 2} f(x)+g(x) = 7$.

#1 Since $\lim_{(x,y)\to(a,b)} f(x,y) = L$, for any $\varepsilon>0$ there is a $\delta_1>0$ so that $\|(x,y)-(a,b)\|<\delta_1$ implies $|f(x,y)-L|<\varepsilon$.

Since $\lim_{(x,y)\to(a,b)} g(x,y) = M$, for any $\varepsilon>0$ there is a $\delta_2>0$ so that $\|(x,y)-(a,b)\|<\delta_2$ implies $|g(x,y)-M|<\varepsilon$.

Take $\delta = \min\{\delta_1,\delta_2\}$. Then $|f(x,y)+g(x,y)-(L+M)| = |(f(x,y)-L)+(g(x,y)-M)|$
$$\leq |f(x,y)-L|+|g(x,y)-M| < \varepsilon+\varepsilon = 2\varepsilon.$$

triangle inequality

Therefore $\lim_{x\to 2}(f(x,y)+g(x,y)) = L+M$.

#2 ~~one way~~ Try different paths.

$y=0$ $\quad \lim_{x\to 0}\frac{x^2\cdot 0}{x^2+0^2} = \lim_{x\to 0} 0 = 0$

$x=0$ $\quad \lim_{y\to 0}\frac{0\cdot y}{0^2+y^2} = \lim_{y\to 0} 0 = 0.$

$y=mx$ $\quad \lim_{x\to 0}\frac{x^2\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0}\frac{x^2\cdot mx}{x^2(1+m^2)} = \lim_{x\to 0}\frac{mx}{1+m^2} = 0$

Since these are all equal, the limit probably exists.

To prove the limit does exist and equal 0, we need to use polar coordinates. Take $x = r\cos\theta$, $y = r\sin\theta$, and take limit as $r\to 0$:

$$\lim_{r\to 0}\frac{(r\cos\theta)^2(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2} = \lim_{r\to 0}\frac{r^3\cos\theta\sin\theta}{r^2(\cancel{\cos^2\theta+\sin^2\theta})} = \underbrace{\lim_{r\to 0} r\cos\theta\sin\theta = 0.}_{\text{use squeeze theorem (see last page)}}$$

Therefore $\boxed{\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2} = 0}$

#2) another way

Squeeze Theorem. Observe that $0 \le x^2 \le x^2+y^2$. Then

only true for $y \ge 0$ $y \cdot 0 \le x^2 y \le (x^2+y^2)y \Rightarrow \frac{0}{x^2+y^2} \le \frac{x^2y}{x^2+y^2} \le \frac{(x^2+y^2)y}{x^2+y^2}$

Incorrect.

$\Rightarrow 0 \le \frac{x^2y}{x^2+y^2} \le y.$

Take the limit: $\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} y$

$0 \le \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} \le 0$

These are the same, so the middle limit must also go to 0

$\Rightarrow \boxed{\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.}$

#3) $f(x,y) = x^3 - 3x^2y + y^3$

$\frac{\partial f}{\partial x} = 3x^2 - 6xy + 0 = \boxed{3x^2 - 6xy}$

$\frac{\partial f}{\partial y} = \boxed{-3x^2 + 3y^2}$

#4) $f(x,y) = \frac{x+y}{y+1}$

only one x term on numerator, so we don't need quotient rule:

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+y}{y+1}\right) = \frac{1}{y+1}\frac{\partial}{\partial x}(x+y) = \frac{1}{y+1}\cdot 1 = \frac{1}{y+1}$

y terms on numerator and denominator, so we need to use quotient rule: $\left(\frac{T}{B}\right)' = \frac{BT' - TB'}{B^2}$

$\frac{\partial f}{\partial y} = \frac{(y+1)\frac{\partial}{\partial y}(x+y) - (x+y)\frac{\partial}{\partial y}(y+1)}{(y+1)^2} = \frac{(y+1)\cdot 1 - (x+y)\cdot 1}{(y+1)^2} = \frac{y+1-x-y}{(y+1)^2}$

$= \boxed{\frac{1-x}{(y+1)^2}}$

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one way $f(a+ht, b) \approx f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$

$\Delta x = a+ht-a \qquad \Delta y = b-b$

$= f(a,b) + f_x(a,b)\,ht$

Plug into limit:

$$\lim_{t\to 0} \frac{\cancel{f(a,b)} + f_x(a,b)\,h\cancel{t} - \cancel{f(a,b)}}{\cancel{t}} = h\,f_x(a,b) \checkmark$$

another way substitution $s = ht$. Then $t = s/h$. Take $s \to 0$:

$$\lim_{t\to 0} \frac{f(a+th,b) - f(a,b)}{t} = \lim_{s\to 0} \frac{f(a+s,b) - f(a,b)}{s/h} \leftarrow \text{rewrite this}$$

$$= \lim_{s\to 0} h \cdot \underbrace{\frac{f(a+s,b) - f(a,b)}{s}}_{\text{this is what gives the } \frac{\partial f}{\partial x} \text{ term}} = h \cdot \lim_{s\to 0} \frac{f(a+s,b) - f(a,b)}{s} = h\,\frac{\partial f}{\partial x}(a,b). \checkmark$$

#2 continued

Want to show $\lim_{r\to 0} r\cos\theta\sin\theta = 0$.

since $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$, then $-1 \le \cos\theta\sin\theta \le 1$

$\Rightarrow -r \le r\cos\theta\sin\theta \le r$

Take the limit of each side: $\lim_{r\to 0} -r \le \lim_{r\to 0} r\cos\theta\sin\theta \le \lim_{r\to 0} 0$

$0 \le \lim_{r\to 0} r\cos\theta\sin\theta \le 0$

$\Rightarrow \lim_{r\to 0} r\cos\theta\sin\theta = 0$ by Squeeze Theorem.