There can be lots of ways to solve an individual problem finding extrema. We will look at a particular example. Let

$$f(x,y) = x + y,$$

and consider the constraint

$$g(x,y) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \quad \Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

(* Note this is closed and bounded)

Lagrange Multipliers

1. Find the extremal values using the method of Lagrange multipliers.

$\nabla f = \langle 1, 1 \rangle$

$\nabla g = \langle \frac{2x}{9}, \frac{y}{8} \rangle$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 1 = \frac{2x}{9}\lambda & (1) \\ 1 = \frac{y}{8}\lambda & (2) \\ \frac{x^2}{9} + \frac{y^2}{16} = 1 & (3) \end{cases}$$

3 equations, 3 unknowns.

(1): Solve for $\lambda \Rightarrow \lambda = \frac{9}{2x}$

Substitute into (2): $1 = \frac{y}{8}\left(\frac{9}{2x}\right) \Rightarrow \frac{9y}{16x} = 1$ Solve for either x or y

$y = \frac{16x}{9}$

Substitute into (3): $16 \cdot 9\left(\frac{x^2}{9} + \frac{\left(\frac{16x}{9}\right)^2}{16}\right) = (1)\, 9 \cdot 16$ Multiply both sides by $9 \cdot 16 = 144$

$16x^2 + 9\left(\frac{16x}{9}\right)^2 = 144$

$\frac{9}{9} \cdot 16x^2 + \frac{256x^2}{9} = 144$

$\left(\frac{144+256}{9}\right)x^2 = 144$

$\frac{400}{9}x^2 = 144$

$x^2 = \frac{144 \cdot 9}{400}$

1 $x = \pm\sqrt{\frac{144 \cdot 9}{400}}$

$= \pm\frac{36}{20}$

$= \pm\frac{9}{5}$

If $x = +\frac{9}{5}$, then

$y = \frac{16}{9}x = \frac{16}{9} \cdot \frac{9}{5} = \frac{16}{5}$

If $x = \frac{-9}{5}$ then

$y = \frac{-16}{5}$

Therefore $\left(\frac{9}{5}, \frac{16}{5}\right), \left(-\frac{9}{5}, -\frac{16}{5}\right)$ are the extrema.

$$\boxed{\begin{array}{ll} f\left(\frac{9}{5}, \frac{16}{5}\right) = 5 & \text{local max} \\ f\left(\frac{-9}{5}, \frac{-16}{5}\right) = -5 & \text{local min} \end{array}}$$

2. Parameterize the curve $g(x, y) = 1$ by considering

$$\langle x(t), y(t)\rangle = \langle 3\cos t, 4\sin t\rangle$$ ← i.e. This is a way to write the given ellipse

with $0 \le t \le 2\pi$. Plugging this into f, find the extrema of the one variable function of t.

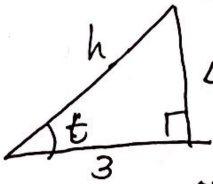
$f(x,y) = x+y$ plug in $x(t), y(t)$

$f(t) = 3\cos t + 4\sin t$ set derivative $=0$

$f'(t) = -3\sin t + 4\cos t = 0$

$4\cos t = 3\sin t$

$\frac{4}{3} = \tan t$



← This is quadrant I

* tan is positive in 2 quadrants (quadrant I and III)

recall that in a right triangle $\tan t = \frac{opp}{adj}$

Use Pythagorean's Theorem to find h: $3^2+4^2=h^2$

$25 = h^2$

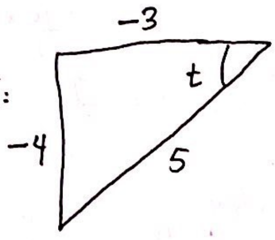
$h=5$

Therefore $\sin t = \frac{4}{5}$ (opp/hyp) and $\cos t = \frac{3}{5}$ (adj/hyp).

Plug back into original function

$\boxed{f(x,y) = 3\left(\frac{3}{5}\right)+4\left(\frac{4}{5}\right)=5}$ (1) (same max as L(x,y))

quadrant III:



In this case, $\boxed{f(x,y) = 3\left(\frac{-3}{5}\right)+4\left(\frac{-4}{5}\right)=-5}$ (1) (same min as L(x,y))