Practice Problems I Solutions

1. Differentiate the function.

(a) $f(x) = e^{x \cos 2x}$

Chain rule

$$f'(x) = \left(e^{x\cos 2x}\right)' = e^{x\cos 2x}(x\cos 2x)' = e^{x\cos 2x}(1\cdot\cos 2x + x\cdot -\sin 2x\cdot 2)$$

$$= \boxed{e^{x\cos 2x}(\cos 2x - 2x\sin 2x)}$$

(b) $f(x) = (x^3 - 2x)4^x$

$$f'(x) = \left((x^3-2x)4^x\right)' = (x^3-2x)'\,4^x + (x^3-2x)(4^x)' = (3x^2-2)4^x + (x^3-2x)4^x\ln 4$$

Product rule

$$= \boxed{4^x\left[(x^3-2x)\ln 4 + 3x^2-2\right]}$$

(c) $f(x) = \log_3(2-5x) = \dfrac{\ln(2-5x)}{\ln 3}$

Change of base

$$\Rightarrow f'(x) = \left(\frac{1}{\ln 3}\ln(2-5x)\right)' = \frac{1}{\ln 3}\,\frac{1}{2-5x}(-5) = \boxed{\frac{-5}{\ln 3(2-5x)}}$$

(d) $f(x) = \ln(x^4 \sin^2 x)$

$$f'(x) = \left(\ln(x^4\sin^2 x)\right)' = \frac{1}{x^4\sin^2 x}(x^4\sin^2 x)' = \frac{4x^3\sin^2 x + x^4\cdot 2\sin x\cos x}{x^4\sin^2 x}$$

chain rule

$$= \boxed{\frac{4x^3\sin^2 x + 2x^4\sin x\cos x}{x^4\sin^2 x}}$$

2. Use logarithmic differentiation to find the derivative of

$$f(x) = \frac{x^4(2x^2+1)^{1/3}}{\cos x}$$

One way

Let $y = \ln f(x) = \ln\left(\dfrac{x^4(2x^2+1)^{1/3}}{\cos x}\right) = \ln(x^4) + \ln\left((2x^2+1)^{1/3}\right) - \ln(\cos x) = 4\ln x + \frac{1}{3}\ln(2x^2+1) - \ln\cos x$

$$y' = 4\cdot\frac{1}{x} + \frac{1}{3}\cdot\frac{1}{2x^2+1}(2x^2+1)' - \frac{1}{\cos x}(\cos x)' = \frac{4}{x} + \frac{4x}{3(2x^2+1)} + \frac{\sin x}{\cos x} = \frac{4}{x} + \frac{4x}{3(2x^2+1)} + \tan x$$

$$f'(x) = f(x)(\ln y)' = \boxed{\frac{x^4(2x^2+1)^{1/3}}{\cos x}\left[\frac{4}{x} + \frac{4x}{3(2x^2+1)} + \tan x\right]}$$

Another way

Take $\ln$ of both sides: $\ln f(x) = \ln\left(\dfrac{x^4(2x^2+1)^{1/3}}{\cos x}\right) = 4\ln x + \frac{1}{3}\ln(2x^2+1) - \ln(\cos x)$

Take derivative of both sides: $\dfrac{1}{f(x)}f'(x) = \dfrac{4}{x} + \dfrac{1}{3}\cdot\dfrac{1}{2x^2+1}\cdot 4x - \dfrac{1}{\cos x}\cdot -\sin x$

$$\frac{f'(x)}{f(x)} = \frac{4}{x} + \frac{4x}{3(2x^2+1)} + \tan x$$

Solve for $f'(x)$: $f'(x) = f(x)\left(\dfrac{4}{x} + \dfrac{4x}{3(2x^2+1)} + \tan x\right) = \boxed{\dfrac{x^4(2x^2+1)^{1/2}}{\cos x}\left(\dfrac{4}{x} + \dfrac{4x}{3(2x^2+1)} + \tan x\right)}$

3. Evaluate the integral.

(a) $\displaystyle\int_0^6 e^{-2x}\,dx$

$$\int_0^6 e^{\overset{u}{-2x}}\,\underset{\frac{du}{-2}}{dx}$$

$u = -2x$

$du = -2\,dx \Rightarrow dx = \dfrac{du}{-2}$

If $x=0$, $u = -2\cdot 0 = 0$

$x = 6$, $u = -2\cdot 6 = -12$

$$\int_0^{-12} e^u\,\frac{du}{-2} = -\frac{1}{2}\,e^u\Big|_0^{-12} = -\frac{1}{2}\left(e^{-12} - e^0\right) = \boxed{-\frac{1}{2}\left(e^{-12}-1\right)}$$

(b) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ $\quad u=\sqrt{x}=x^{1/2}$

$du=\frac{1}{2}x^{-1/2}dx=\frac{1}{2\sqrt{x}}dx \Rightarrow 2du=\frac{1}{\sqrt{x}}dx$

$\int e^u \cdot 2du = 2e^u + C = \boxed{2e^{\sqrt{x}}+C}$

(c) $\int \frac{2x^3}{x^4+1}dx$

$\int \frac{2x^3dx}{x^4+1}$ $\quad u=x^4+1$

$du=4x^3dx \Rightarrow \frac{du}{2}=2x^3dx$

$\int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|+C = \boxed{\frac{1}{2}\ln|x^4+1|+C}$

(d) $\int \tan x\, dx$

$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x}$ $\quad u=\cos x$

$du=-\sin x dx \Rightarrow -du=\sin x dx$

$=\int \frac{-du}{u} = -\int \frac{1}{u}du = -\ln|u|+C = \boxed{-\ln|\cos x|+C}$

4. Evaluate the following limits.

(a) $\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}} = \lim_{x\to\infty}\frac{e^x-\frac{1}{e^x}}{e^x+\frac{1}{e^x}}\cdot\frac{e^x}{e^x}$ (Multiply num. and denom. by e^x) $=\lim_{x\to\infty}\frac{e^{2x}-1}{e^{2x}+1} = \frac{\infty}{\infty}$ indeterminate

$\overset{LH}{=} \lim_{x\to\infty}\frac{(e^{2x}-1)'}{(e^{2x}+1)'} = \lim_{x\to\infty}\frac{2e^{2x}}{2e^{2x}} = \lim_{x\to\infty} 1 = 1$

(b) $\lim_{x\to 0}\ln(\cos x) = \ln\left(\lim_{x\to 0}\cos x\right) = \ln(\cos 0) = \ln(1) = 0$

ln is continuous

(c) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = (1+0)^\infty = 1^\infty$ indeterminate. Let $L=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. Take ln of both sides...

$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \infty\cdot\ln 1 = \infty\cdot 0$, still indeterminate

ln is continuous

$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{\frac{-1}{x^2+x}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{x^2}{x^2+x} = \frac{\infty}{\infty}$ indeterminate

$\overset{LH}{=}\lim_{x\to\infty}\frac{2x}{2x+1} = \frac{\infty}{\infty}$ indeterminate

$=\lim_{x\to\infty}\frac{2}{2} = \lim_{x\to\infty} 1 = 1$

$\ln L = 1 \Rightarrow L = e^1 = e$

(d) $\lim_{x\to\infty}(\ln(1+x)-\ln(3+x)) = \infty-\infty$ indeterminate

$=\lim_{x\to\infty}\ln\left(\frac{1+x}{3+x}\right) = \ln\left(\lim_{x\to\infty}\frac{1+x}{3+x}\right) = \ln\left(\frac{\infty}{\infty}\right)$ inside is indeterminate, use L'Hôpital's rule on just the inside

ln is continuous

$\overset{LH}{=}\ln\left(\lim_{x\to\infty}\frac{(1+x)'}{(3+x)'}\right) = \ln\left(\lim_{x\to\infty}\frac{1}{1}\right) = \ln(1) = 0$

(e) $\lim\limits_{x\to 0} \dfrac{6^x - 3^x}{x} = \dfrac{6^0-3^0}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}$ indeterminate

$\overset{LH}{=} \lim\limits_{x\to 0} \dfrac{6^x \ln 6 - 3^x \ln 3}{1} = 6^0 \ln 6 - 3^0 \ln 3 = \ln 6 - \ln 3 = \ln \frac{6}{3} = \ln 2$

(f) $\lim\limits_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x} \right) = \dfrac{1}{1-1} - \dfrac{1}{\ln 1} = \dfrac{1}{0} - \dfrac{1}{0} = \infty - \infty$ indeterminate

Common denominator

$\lim\limits_{x\to 1} \left(\dfrac{x \ln x - (x-1)}{(x-1)\ln x} \right) = \dfrac{1\cdot 0 - (1-1)}{(1-1)\cdot 0} = \dfrac{0}{0}$ still indeterminate

$\overset{LH}{=} \lim\limits_{x\to 1} \left(\dfrac{1\cdot \ln x + x \cdot \frac{1}{x} - 1}{\ln x + (x-1)\cdot \frac{1}{x}} \right) = \lim\limits_{x\to 1} \left(\dfrac{\ln x}{\ln x + \frac{x-1}{x}} \right)\dfrac{\cdot x}{\cdot x} = \lim\limits_{x\to 1} \dfrac{x \ln x}{x \ln x + x - 1} = \dfrac{0}{0}$ still indeterminate

$\overset{LH}{=} \lim\limits_{x\to 1} \dfrac{\ln x + x\cdot\frac{1}{x}}{\ln x + x\cdot\frac{1}{x} + 1} = \lim\limits_{x\to 1} \dfrac{\ln x + 1}{\ln x + 2} = \dfrac{\ln 1 + 1}{\ln 1 + 2} = \frac{1}{2}$

5. Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = e^{1/n}$ $\qquad \lim\limits_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty} \frac{1}{n}} = e^0 = 1$ converges

e^x is continuous

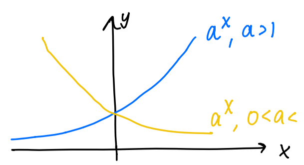
(b) $a_n = n \sin\left(\dfrac{1}{n}\right)$ $\qquad \lim\limits_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \infty \cdot \sin 0 = \infty \cdot 0$ indeterminate

Change to functions: $\lim\limits_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim\limits_{x\to\infty} x \sin\left(\frac{1}{x}\right) = \lim\limits_{x\to\infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{\cos\left(\frac{1}{x}\right)\cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}}$

$= \cos\left(\lim\limits_{x\to\infty} \frac{1}{x} \right) = \cos(0) = 1$ converges

cos is continuous

(c) $a_n = 1 - (0.2)^n$ $\qquad \lim\limits_{x\to\infty} a^x = \begin{cases} \infty & \text{if } a>1 \\ 0 & \text{if } 0<a<1 \end{cases}$



$\Rightarrow \lim\limits_{n\to\infty} \left(1 - 0.2^n\right) = 1 - 0 = 1$ converges

(d) $a_n = n^2 e^{-n}$ $\qquad \lim\limits_{n\to\infty} \dfrac{n^2}{e^n} = \dfrac{\infty}{\infty}$ indeterminate, change to x's so we can use L'Hopital's rules

$\lim\limits_{n\to\infty} \dfrac{n^2}{e^n} = \lim\limits_{x\to\infty} \dfrac{x^2}{e^x} \overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} = \dfrac{\infty}{\infty}$

$\overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$ converges

(e) $a_n = \dfrac{(-1)^{n-1} n}{n^2+1}$ $\qquad$ One way Look at $\lim\limits_{n\to\infty} |a_n|$.

$\lim\limits_{n\to\infty} \left| \dfrac{(-1)^{n-1} n}{n^2+1} \right| = \lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = \lim\limits_{x\to\infty} \dfrac{x}{x^2+1} \overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{1}{2x} = 0$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ converges

Another way Squeeze Theorem.

$\dfrac{-n}{n^2+1} \le \dfrac{(-1)^{n-1} n}{n^2+1} \le \dfrac{n}{n^2+1}$

$\cancelto{0}{\lim\limits_{n\to\infty} \dfrac{-n}{n^2+1}} \le \lim\limits_{n\to\infty} \dfrac{(-1)^{n-1} n}{n^2+1} \le \cancelto{0}{\lim\limits_{n\to\infty} \dfrac{n}{n^2+1}}$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ converges

(f) $a_n = \dfrac{\cos^2 n}{2^n}$ Squeeze Theorem. $0 \le \cos^2(n) \le 1 \Rightarrow \dfrac{0}{2^n} \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$

$\Rightarrow \cancel{\lim_{n\to\infty} 0} \to 0 \le \lim_{n\to\infty} \dfrac{\cos^2 n}{2^n} \le \cancel{\lim_{n\to\infty} \dfrac{1}{2^n}} \to 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$ converges

(g) $a_n = \dfrac{n^n}{n!}$

$a_n = \dfrac{n^n}{n!} = \dfrac{n \quad n \quad n \cdots n \cdot n \cdot n}{n\,(n-1)\,(n-2) \cdots 3 \cdot 2 \cdot 1}$

$= \underbrace{\dfrac{n}{n}}_{=1} \cdot \underbrace{\dfrac{n}{n-1}}_{\geq 1} \cdot \underbrace{\dfrac{n}{n-2}}_{\geq 1} \cdots \underbrace{\dfrac{n}{3}}_{\geq 1} \cdot \underbrace{\dfrac{n}{2}}_{\geq 1} \cdot n \geq 1 \cdot 1 \cdots 1 \cdot 1 \cdot n = n$

So $\dfrac{n^n}{n!} \geq n \Rightarrow \lim_{n\to\infty} \dfrac{n^n}{n!} \geq \lim_{n\to\infty} n = \infty \Rightarrow \lim_{n\to\infty} a^n = \infty$ diverges

(h) $a_n = \dfrac{e^n}{n!}$

$a_n = \dfrac{e \quad e \quad e \cdots e \cdot e \cdot e}{n\,\underbrace{(n-1)}_{\le 1}\,\underbrace{(n-2)}_{\le 1} \cdots \underbrace{3}_{\le 1} \cdot 2 \cdot 1} \le \dfrac{e}{n}\, 1 \cdot 1 \cdots 1 \cdot \dfrac{e}{2} \cdot e = \dfrac{e^2}{2n}$

$0 \le a_n \le \dfrac{e^2}{2n} \Rightarrow \lim_{n\to\infty} 0 \le \lim_{n\to\infty} a_n \le \cancel{\lim_{n\to\infty} \dfrac{e^2}{2n}} \to 0 \Rightarrow \lim_{n\to\infty} a_n = 0$ by squeeze theorem

converges

*6. Determine whether the series is convergent or divergent. State what test(s) you used to come to your conclusion.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^n}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{3^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges, by geometric series

geometric series w/ $r = \frac{1}{2}$ converges

geometric series w/ $r = \frac{3}{2}$ diverges

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ Let $a_n = \dfrac{e^n}{n^2}$ $\quad \lim_{n\to\infty} \dfrac{e^n}{n^2} = \dfrac{\infty}{\infty}$ indeterminate

$= \lim_{x\to\infty} \dfrac{e^x}{x^2} \overset{LH}{=} \lim_{x\to\infty} \dfrac{e^x}{2x} \overset{LH}{=} \lim_{x\to\infty} \dfrac{e^x}{2} = \infty \neq 0$ divergent by Divergence Test

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{0.85}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ p-series w/ $p = 0.85$ divergent by p-series

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{10^n}$ $\quad \dfrac{1+\sin n}{10^n} \le \dfrac{1+1}{10^n} = \dfrac{2}{10^n}$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{2}{10^n} = 2\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$ is convergent geometric series

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{10^n}$ converges by Direct Comparison

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} = \sum_{n=1}^{\infty}\left(\frac{n}{n\sqrt{n}} + \frac{1}{n\sqrt{n}}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} + \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ diverges by p-series

p-series w/ $p = 1/2$ divergent

p-series w/ $p = 3/2$ convergent

(f) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right)$ $\quad \lim_{n\to\infty} \cos\left(\dfrac{\pi}{n}\right) = \cos\left(\lim_{n\to\infty} \dfrac{\pi}{n}\right) = \cos(0) = 1 \neq 0$ diverges by divergence test

cos is continuous

(g) $\sum_{n=3}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)$ Write out some partial sums.

$$\sum_{n=3}^{4}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)=\left(\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{1}{5}-\frac{1}{3}\right)$$

$$\sum_{n=3}^{5}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)=\left(\cancel{\frac{1}{4}}-\underline{\frac{1}{2}}\right)+\left(\underline{\frac{1}{5}}-\underline{\frac{1}{3}}\right)+\left(\underline{\frac{1}{6}}-\cancel{\frac{1}{4}}\right)$$

$$\sum_{n=3}^{6}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)=\left(\cancel{\frac{1}{4}}-\underline{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{5}}-\underline{\frac{1}{3}}\right)+\left(\underline{\frac{1}{6}}-\cancel{\frac{1}{4}}\right)+\left(\underline{\frac{1}{7}}-\cancel{\frac{1}{5}}\right)$$ terms cancel out!

$$\sum_{n=3}^{7}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)=\left(\cancel{\frac{1}{4}}-\underline{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{5}}-\underline{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{6}}-\cancel{\frac{1}{4}}\right)+\left(\underline{\frac{1}{7}}-\cancel{\frac{1}{5}}\right)+\left(\underline{\frac{1}{8}}-\cancel{\frac{1}{6}}\right)$$

these terms will never cancel out

notice the last two terms remain

Now write for a general partial sum:

$$\sum_{n=1}^{N}\left(\frac{1}{n+1}-\frac{1}{n}\right)=\left(\cancel{\frac{1}{4}}-\frac{1}{2}\right)+\left(\cancel{\frac{1}{5}}-\frac{1}{3}\right)+\left(\cancel{\frac{1}{6}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\frac{1}{N-1+1}-\cancel{\frac{1}{N-1-1}}\right)+\left(\frac{1}{N+1}-\cancel{\frac{1}{N-1}}\right)$$

$$=-\frac{1}{2}-\frac{1}{3}+\frac{1}{N}+\frac{1}{N+1}$$

Then $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right)=\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n+1}-\frac{1}{n}\right)=\lim_{N\to\infty}\left(-\frac{1}{2}-\frac{1}{3}+\frac{1}{N}+\frac{1}{N+1}\right)=-\frac{1}{2}-\frac{1}{3}+0+0=-\frac{5}{6}$