WEEK 3 Sequence Examples

① If $a_1=5$, $a_{n+1}=\sqrt{4a_n+3}$, Find $\lim_{n\to\infty} a_n$.

It turns out this sequence is bounded below and decreasing $\Rightarrow$ limit exists.

Let $L=\lim_{n\to\infty} a_n$.

$\lim_{n\to\infty} a_{n+1} = \sqrt{4\lim_{n\to\infty} a_n+3}$

$L=\sqrt{4L+3}$

$L^2=4L+3$

$L^2-4L-3=0.$

$L=\dfrac{4\pm\sqrt{(-4)^2-4(1)(-3)}}{2(1)} = \dfrac{4\pm\sqrt{28}}{2} = \dfrac{4\pm 2\sqrt{7}}{2} = 2\pm\sqrt{7}$

$\boxed{L=2+\sqrt{7}}$

② Write a_n for the following sequences: (i) $1, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \dots$

(ii) $1, 2, 6, 24, 120, \dots$

(iii) $\frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots$

Which of these converge?

(i) $a_1=1$ $\quad a_2=\frac{2}{3}$ $\quad a_3=\frac{4}{3}=\frac{2^2}{3}$ $\quad a_4=\frac{8}{3}=\frac{2^3}{3}$

this doesn't really fit into the pattern, so we can write

these three have the pattern $a_n=\frac{2^{n-1}}{3}$ for $n\geq 2$

$\boxed{a_1=1,\ a_n=\frac{2^{n-1}}{3} \text{ for } n\geq 2}$ Does not converge because $\lim_{n\to\infty}\frac{2^{n-1}}{3}=\infty$.

(ii) $a_1=1$ $\quad a_2=2=2\cdot 1$ $=2!$ $\quad a_3=6=3\cdot 2\cdot 1$ $=3!$ $\quad a_4=24=4\cdot 3\cdot 2\cdot 1$ $=4!$ $\quad a_5=120=5\cdot 4\cdot 3\cdot 2\cdot 1$ $=5!$

$\boxed{a_n=n!}$ Does not converge because $\lim_{n\to\infty} n!=\infty$

(iii) $a_1=\frac{4}{9}$ $\quad a_2=\frac{8}{27}=\frac{2}{3}\cdot\frac{4}{9}$ $\quad a_3=\frac{16}{81}=\left(\frac{2}{3}\right)^2\frac{4}{9}$, ... $\quad \boxed{a_n=\frac{4}{9}\left(\frac{2}{3}\right)^{n-1}}$ or $\boxed{a_n=\left(\frac{2}{3}\right)^{n+1}}$

Does converge because $\frac{2}{3}<1$, $\lim_{n\to\infty}\left(\frac{2}{3}\right)^{n+1}=0$.