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Math 31B Worksheet
Week 4

1. Write each of the following sums in summation notation (i.e., with a $\sum$):

   (a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$ (finite sum) $\quad \sum_{n=1}^{5} \frac{1}{2n}$

   (b) $\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\frac{1}{7\cdot 8}$ (finite sum) $\quad \sum_{n=3}^{7} \frac{1}{n(n+1)}$

   (c) $\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots \quad \sum_{n=1}^{\infty} \frac{1}{n!}$

   (d) $9-3+1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots \quad \sum_{n=0}^{\infty} 9\left(\frac{-1}{3}\right)^n$

2. For each of the following series, compute the partial sums S_1, S_2, S_3, and S_4.

   (a) $4+2+1+\frac{1}{2}+\frac{1}{4}+\cdots$

   $S_1 = 4$
   $S_2 = 4+2 = 6$
   $S_3 = 4+2+1 = 7$
   $S_4 = 4+2+1+\frac{1}{2} = \frac{15}{2}$

   (b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

   $S_1 = \sum_{n=1}^{1} \frac{1}{n(n+1)} = \frac{1}{1(2)} = \frac{1}{2}$

   $S_2 = \sum_{n=1}^{2} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} = \frac{1}{2}+\frac{1}{6} = \frac{2}{3}$

   $S_3 = \sum_{n=1}^{3} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{2}{3}+\frac{1}{12}$
   $= \frac{3}{4}$

   $S_4 = \sum_{n=1}^{4} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}$
   $= \frac{3}{4}+\frac{1}{20} = \frac{4}{5}$

3. Consider a series $\sum_{n=1}^{\infty} a_n$, and its partial sums $S_N = \sum_{n=1}^{N} a_n$.

   (a) Complete the following definition: the series *converges* if and only if

   the limit of the partial sums converges

   (b) True/False: If $\lim_{n\to\infty} a_n = 0$, then the series converges.

   False. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges even though $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Another example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

   (c) True/False: If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

   True. This is called the divergence test.

For the True/False questions above: If true, what theorem tells you it's true? If false, give an example that demonstrates why it's false.

4. In each part of this problem, you are given a geometric series. For each one, do these steps: i. Identify the "common ratio" r between the terms. ii. Identify c, the *first term* of the series. iii. Does the series converge, and if so, what value does it converge to?

(a) $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$

i. $r = \frac{1}{2}$

ii. $c = 4$

iii. Diverges? Or converges to ? converges since $|r| = |\frac{1}{2}| = \frac{1}{2} < 1$.

converges to $\frac{4}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8$

(b) $\frac{1}{2} - \frac{3}{4} + \frac{9}{8} - \frac{27}{16} + \cdots$

i. $r = \frac{-3}{2}$

ii. $c = \frac{1}{2}$

iii. Diverges? Or converges to ? Diverges since $|r| = |-\frac{3}{2}| = \frac{3}{2} \geq 1$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{7 \cdot 2^n}{3^n} = \underbrace{\frac{7 \cdot 2^1}{3^1}}_{\text{first term}} + \frac{7 \cdot 2^2}{3^2} + \cdots = \sum_{n=0}^{\infty} \frac{14}{3}\left(\frac{2}{3}\right)^n$

i. $r = \frac{2}{3}$

ii. $c = \frac{14}{3}$

iii. Diverges? Or converges to ? $|r| = \frac{2}{3} < 1$, converges to $\frac{14/3}{1-2/3} = \frac{14/3}{1/3} = 14$

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\pi}{e}\right)^n = \underbrace{\left(\frac{\pi}{e}\right)^0}_{\text{first term}} + \left(\frac{\pi}{e}\right)^1 + \cdots$

i. $r = \frac{\pi}{e}$

ii. $c = 1$

iii. Diverges? Or converges to ? $|r| = \frac{\pi}{e} > 1$, diverges

(e) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{5}{3^n} = \sum_{n=2}^{\infty} 5\left(\frac{-1}{3}\right)^n = \underbrace{5\left(\frac{-1}{3}\right)^2}_{\text{first term}} + 5\left(\frac{-1}{3}\right)^3 + \cdots = \sum_{n=0}^{\infty} \frac{5}{9}\left(\frac{-1}{3}\right)^n$

i. $r = \frac{-1}{3}$

ii. $c = \frac{5}{9}$

iii. Diverges? Or converges to ? $|r| = \frac{1}{3} < 1$, converges to $\frac{5/9}{1-(-1/3)} = \frac{5/9}{4/3} = \frac{5}{9} \cdot \frac{3}{4} = \frac{5}{12}$

5. For this problem, consider the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\big(\ln(n)\big)^{1.1}}$, and its partial sums S_N.

(a) Compute the integral $\displaystyle\int_{x=2}^{x=N} \frac{1}{x\big(\ln(x)\big)^{1.1}}\,dx$. (Your answer will be in terms of N.)

Then compute the limit of this as $N \to \infty$.

$$\int_{x=2}^{N} \frac{1}{x(\ln x)^{1.1}}\,dx = \int_{\ln 2}^{\ln N} \frac{1}{u^{1.1}}\,du = \int_{\ln 2}^{\ln N} u^{-1.1}\,du = \frac{u^{-0.1}}{-0.1}\Bigg|_{\ln 2}^{\ln N}$$

du

$u = \ln x$

$du = \frac{1}{x}\,dx$

when $x=2$, $u = \ln 2$

$x = N$ $u = \ln N$

$$= \frac{(\ln N)^{-0.1}}{-0.1} - \frac{(\ln 2)^{-0.1}}{-0.1} = \frac{1}{-0.1(\ln N)^{0.1}} + \frac{1}{0.1(\ln 2)^{0.1}}$$

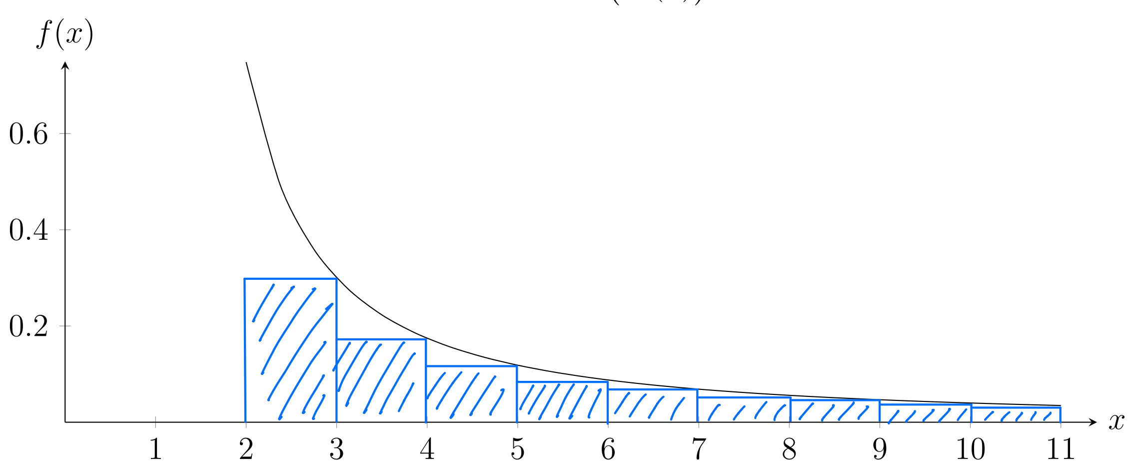
Take limit as $N \to \infty$:

$$\lim_{N\to\infty}\left(\frac{1}{-0.1(\ln N)^{0.1}}\right) + \frac{1}{0.1(\ln 2)^{0.1}} = 0 + \frac{1}{0.1(\ln 2)^{0.1}} = \frac{1}{0.1(\ln 2)^{0.1}}$$

← we got a finite #, so we say the integral converges

(*Note: $(\ln N)^{0.1} \neq 0.1 \ln N$,
$\ln(N^{0.1}) = 0.1 \ln N$)

(b) The graph below is of the function $f(x) = \dfrac{1}{x\big(\ln(x)\big)^{1.1}}$ that you just integrated.



Note that the function is ***positive***, ***decreasing***, and ***continuous***.

On the graph, sketch rectangles that represent a Riemann sum for the integral

$$\int_{2}^{10} f(x)\,dx$$

Use sub-intervals of size $\Delta x = 1$, and use the *right endpoint* of each sub-interval for your sample points.

(c) Explain how the Riemann sum you just sketched is equal to a partial sum S_N for the series $\sum_{n=3}^{\infty} \frac{1}{n\big(\ln(n)\big)^{1.1}}$ (and for what value of N?)

Riemann sum means to add up the area of the rectangles, general form: $\sum_{n=1}^{N} \underbrace{\Delta x}_{\text{width of rectangles}} \underbrace{f(x_n)}_{\text{height of rectangles}}$

In (b), the width of the rectangles is $\Delta x = 1$, the height is $f(n) = \frac{1}{n(\ln n)^{1.1}}$

starting w/ $n=3$: $\sum_{n=3}^{11} \frac{1}{n(\ln n)^{1.1}}$

$N=11$ because that is the last endpoint we used

(d) Which is larger/smaller, the area represented by the Riemann sum, or the area represented by the integral? Express this as an inequality relating S_N to an integral.

The Riemann sum (aka partial sum) is an underestimate to the integral:

$$S_N \leq \int_2^N f(x)\,dx$$

(e) Explain how the results of parts (a) and (d) show that the series converges or diverges.

Since all the partial sums are smaller than the integral, then the infinite series will be smaller than the integral:

$$\lim_{N\to\infty} \sum_{n=3}^{N} \frac{1}{n(\ln n)^{1.1}} \leq \lim_{N\to\infty} \int_2^N \frac{1}{x(\ln x)^{1.1}}\,dx$$

$$\Rightarrow \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{1.1}} \leq \int_2^{\infty} \frac{1}{x(\ln x)^{1.1}}\,dx$$

In (a), the integral converges $\Rightarrow$ the series converges.

* Integral Test: If $f(x)$ is decreasing, positive, and $\lim_{x\to\infty} f(x) = 0$, $a_n = f(n)$.

① If $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

② If $\int_1^{\infty} f(x)\,dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.