PRACTICE PROBLEMS I SOLUTIONS

1. Evaluate the following integrals:

(a) $\int_1^4 \int_1^2 \left(\frac{x}{y}+\frac{y}{x}\right) dydx$

$$= \int_1^4 \left(x\ln|y| + \frac{y^2}{2x} \right)\Big|_{y=1}^{2} dx = \int_1^4 \left(x(\ln 2 - \cancel{\ln 1}^{\,0}) + \frac{1}{2x}(4-1)\right) dx$$

$$= \int_1^4 \left(x\ln 2 + \frac{3}{2x}\right) dx = \frac{x^2}{2}\ln 2 + \frac{3}{2}\ln|x| \Big|_1^4 = \frac{\ln 2}{2}(16-1) + \frac{3}{2}(\ln 4 - \ln 1)$$

$$= \boxed{\frac{15}{2}\ln 2 + \frac{3}{2}\ln 4} = \frac{15}{2}\ln 2 + \frac{3}{2}\ln 2^2 = \frac{15}{2}\ln 2 + 3\ln 2 = \frac{21}{2}\ln 2$$

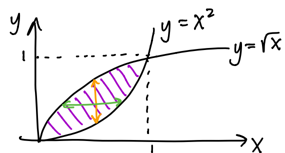
(b) $\int_0^1 \int_0^1 \sqrt{s+t}\, dsdt$

$$= \int_0^1 \frac{2}{3}(s+t)^{3/2}\Big|_{s=0}^{1} dt = \int_0^1 \frac{2}{3}\left((1+t)^{3/2} - t^{3/2}\right) dt$$

$$= \frac{2}{3}\left[\frac{2}{5}(1+t)^{5/2} - \frac{2}{5}t^{5/2}\right]_{t=0}^{1} = \frac{4}{15}\left[(1+t)^{5/2} - t^{5/2}\right]_{t=0}^{1}$$

$$= \frac{4}{15}\left[2^{5/2} - 1^{5/2} - (1^{5/2} - 0^{5/2})\right] = \frac{4}{15}(2^{5/2}-2)$$

(c) $\iint_D (x+y)dA$ where D is bounded by $y=\sqrt{x}, y=x^2$



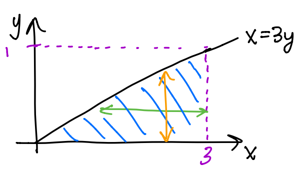
Can use either $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$ orange OR $0 \le y \le 1$, $y^2 \le x \le \sqrt{y}$ green

$$\int_0^1\int_{x^2}^{\sqrt{x}} (x+y)\, dy dx = \int_0^1 \left(xy + \frac{y^2}{2}\right)\Big|_{y=x^2}^{\sqrt{x}} dx = \int_0^1 x(\sqrt{x}-x^2) + \frac{1}{2}(x-x^4)\, dx = \int_0^1 x^{3/2} - x^3 + \frac{1}{2}x - \frac{1}{2}x^4\, dx$$

$$= \frac{2}{5}x^{5/2} - \frac{1}{4}x^4 + \frac{1}{4}x^2 - \frac{1}{10}x^5 \Big|_0^1 = \frac{2}{5} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{1}{10} = \frac{3}{10}$$

2. Evaluate the integral by reversing the order of integration:

$$\int_0^1 \int_{3y}^3 e^{x^2} dxdy$$

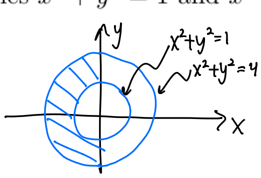


Given $0 \le y \le 1$, $3y \le x \le 3$ $\Rightarrow$ $0 \le x \le 3$, $0 \le y \le \frac{x}{3}$

$$\int_0^3\int_0^{x/3} e^{x^2}\, dy dx = \int_0^3 ye^{x^2}\Big|_{y=0}^{x/3} dx = \int_0^3 \frac{x}{3}e^{x^2} dx \qquad u = x^2,\ du = 2x dx \rightarrow \frac{du}{2} = x dx$$

$$= \int_0^9 \frac{1}{3}e^u \frac{du}{2} = \frac{1}{6}e^u\Big|_0^9 = \boxed{\frac{1}{6}(e^9-1)}$$

3. Evaluate $\iint_R (x+y)dA$ where R is the region that lies to the left of the y-axis between the circles $x^2+y^2=1$ and $x^2+y^2=4$.



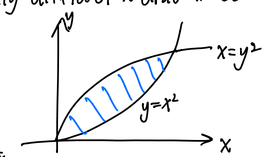
Polar coordinates $1 \le r \le 2$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ $\quad x = r\cos\theta$, $y = r\sin\theta$ $\quad dA = r\,dr\,d\theta$

$$\iint_D (x+y)dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta) r\, dr d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 r^2(\cos\theta+\sin\theta)\, dr d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \frac{r^3}{3}\Big|_{r=1}^{2} (\cos\theta + \sin\theta)\, d\theta = \frac{7}{3}(\sin\theta - \cos\theta)\Big|_{\pi/2}^{3\pi/2} = \frac{7}{3}(-1+0-(1+0)) = -\frac{14}{3}$$

4. Evaluate $\iiint_E xy\,dV$ where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$, and the planes $z = 0$ and $z = x + y$.

Really difficult to draw in 3D. "Bounded by planes $z=0$ and $z=x+y$" $\Rightarrow$ $0 \le z \le x+y$



Can use either $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$ or $0 \le y \le 1$, $y^2 \le x \le y$

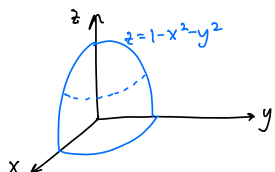
$$\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y} xy\,dz\,dy\,dx = \int_0^1\int_{x^2}^{\sqrt{x}} xyz\Big|_{z=0}^{x+y} dy\,dx = \int_0^1\int_{x^2}^{\sqrt{x}} xy(x+y)\,dy\,dx = \int_0^1\int_{x^2}^{\sqrt{x}} (x^2y+xy^2)\,dy\,dx$$

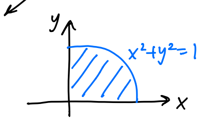
$$= \int_0^1 \left(x^2\frac{y^2}{2}+x\frac{y^3}{3}\right)\Big|_{y=x^2}^{\sqrt{x}} dx = \int_0^1 \left(x^2\cdot\frac{x}{2} + x\frac{x^{3/2}}{3} - \left(x^2\frac{x^4}{2}+x\cdot\frac{x^6}{3}\right)\right)dx$$

$$= \int_0^1 \left(\tfrac12 x^3+\tfrac13 x^{5/2}-\tfrac12 x^6-\tfrac13 x^7\right)dx = \tfrac18 x^4+\tfrac{2}{21}x^{7/2}-\tfrac{1}{12}x^7-\tfrac{1}{24}x^8\Big|_0^1$$

$$= \boxed{\tfrac18+\tfrac{2}{21}-\tfrac{1}{12}-\tfrac{1}{24}}$$

5. Evaluate $\iiint_E (x^3 + xy^2)dV$ where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.





First octant means $x,y,z \ge 0$. We have $0 \le z \le 1-x^2-y^2$

Where does $z=1-x^2-y^2$ intersect w/ $z=0$? $0=1-x^2-y^2 \Rightarrow x^2+y^2=1$.

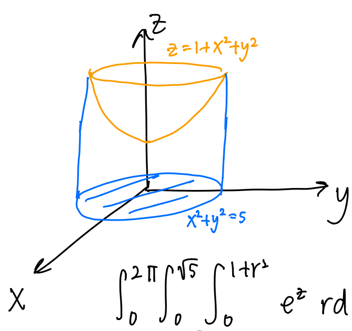
Use cylindrical coordinates: $0\le r\le 1$, $0 \le z \le 1-r^2$, $0\le\theta\le\frac{\pi}{2}$, $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$, $dV=r\,dz\,dr\,d\theta$

$$\iiint_E x(x^2+y^2)dV = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r\cos\theta\,(r^2)\cdot r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta\,dz\,dr\,d\theta$$

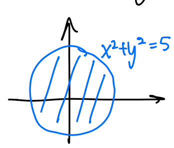
$$= \int_0^{\pi/2}\int_0^1 zr^4\cos\theta\Big|_{z=0}^{1-r^2} dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^4(1-r^2)\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 (r^4-r^6)\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\left(\tfrac15 r^5-\tfrac17 r^6\right)\Big|_{r=0}^1\cos\theta\,d\theta = \int_0^{\pi/2}\left(\tfrac15-\tfrac17\right)\cos\theta\,d\theta = \tfrac{2}{35}\sin\theta\Big|_0^{\pi/2} = \tfrac{2}{35}$$

6. Evaluate $\iiint_E e^z dV$ where E is enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane.



z is bounded by the xy-plane and the paraboloid: $0 \le z \le 1+x^2+y^2$



Change to cylindrical: $0\le r\le\sqrt5$, $0\le\theta\le 2\pi$, $0\le z\le 1+r^2$

since $x=r\cos\theta$, $y=r\sin\theta \Rightarrow x^2+y^2=r^2$

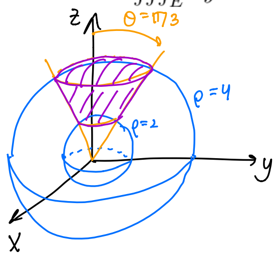
$$\int_0^{2\pi}\int_0^{\sqrt5}\int_0^{1+r^2} e^z\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt5} re^z\Big|_{z=0}^{1+r^2} dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt5}(re^{1+r^2}-r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt5} re^{1+r^2}dr\,d\theta - \int_0^{2\pi}\int_0^{\sqrt5} r\,dr\,d\theta = \int_0^{2\pi}\int_1^6 \tfrac12 e^u\,du\,d\theta - \int_0^{2\pi}\tfrac{r^2}{2}\Big|_0^{\sqrt5}d\theta$$

$u=1+r^2$
$du=2r\,dr$

$$= \int_0^{2\pi}\tfrac12 e^u\Big|_{u=1}^6 d\theta - \int_0^{2\pi}\tfrac52\,d\theta = \int_0^{2\pi}\left(\tfrac12(e^6-e)-\tfrac52\right)d\theta = 2\pi\left[\tfrac12(e^6-e)-\tfrac52\right] = \boxed{\pi(e^6-e-5)}$$

7. Evaluate $\iiint_E xyz\,dV$ where E lies between the spheres $\rho = 2$ and $\rho = 4$ and the cone $\phi = \frac{\pi}{3}$.



spherical coordinates:

$2 \le \rho \le 4$

$0 \le \phi \le \pi/3$

$0 \le \theta \le 2\pi$

$x = \rho\cos\theta\sin\phi$ $\quad z = \rho\cos\phi$

$y = \rho\sin\theta\sin\phi$ $\quad dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_2^4 (\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_2^4 \rho^5 \sin\theta\cos\theta\sin^3\phi\cos\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{6}\rho^6\Big|_{\rho=2}^{4} \sin^3\phi\cos\phi\sin\theta\cos\theta\, d\theta\, d\phi = \frac{4^6-2^6}{6}\int_0^{2\pi}\int_0^{\sqrt{3}/2} u^3\sin\theta\cos\theta\, du\, d\phi$$

$u = \sin\phi$

$du = \cos\phi\, d\phi$

$$= \frac{4^6-2^6}{6}\int_0^{2\pi} \frac{u^4}{4}\Big|_0^{\sqrt{3}/2} \sin\theta\cos\theta\, d\theta = \frac{4^6-2^6}{6}\cdot\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^4 \int_0^0 v\, dv = 0$$

$v = \sin\theta \quad dv = \cos\theta\, d\theta$