PP2 solutions
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1. Determine whether the series is convergent or divergent. State what test(s) you used to come to your conclusion.

(a) $\sum_{n=1}^{\infty} \frac{1+\sin n}{10^n}$

Direct comparison: $\sin(n) \le 1$

$\frac{1+\sin(n)}{10^n} \le \frac{1+1}{10^n} = \frac{2}{10^n}$

$\sum_{n=1}^{\infty} \frac{2}{10^n} = 2\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$ geometric, $|r| = \left|\frac{1}{10}\right| < 1$, converges. $\Rightarrow \sum_{n=1}^{\infty} \frac{1+\sin n}{10^n}$ converges by direct comparison

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{10^n}$

Ratio Test: $L = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)}{10^{n+1}} \cdot \frac{10^n}{n(-1)^n}\right| = \lim_{n\to\infty} \frac{10^n}{10^{n+1}} \cdot \frac{n+1}{n} = \lim_{n\to\infty} \frac{1}{10}\left(1+\frac{1}{n}\right) = \frac{1}{10} < 1$

converges by ratio test

(c) $\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$

Ratio test: $L = \lim_{n\to\infty} \left|\frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n}\right| = \lim_{n\to\infty} \left|\frac{(-10)^{n+1}}{(-10)^n} \cdot \frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(-10)^n}(-10)}{\cancel{(-10)^n}} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}}\right|$

$= \lim_{n\to\infty} \frac{10}{n+1} = 0 < 1$ converges by ratio test

(d) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

Alternating Series Test: $b_n = \frac{1}{n \ln n}$ $\quad \lim_{n\to\infty} \frac{1}{n \ln n} = 0$ ✓

Decreasing? $f(x) = \frac{1}{x\ln x} = (x\ln x)^{-1}$, $f'(x) = -(x\ln x)^{-2}\left[1\cdot \ln x + x\cdot\frac{1}{x}\right] < 0$ ✓

converges by alternating series test

(e) $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

Root test: $L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2+1}{2n^2+1}\right|^n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2} < 1$

converges by root test

(f) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

Limit comparison: compare w/ $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$, which converges since it is a p-series w/ p=2

$\lim_{n\to\infty} \frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{n^3+1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^3}{n^3+1} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}} = 1 \Rightarrow \sum \frac{n}{n^3+1}$ does same thing

converges by limit comparison

(g) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Integral test: $\frac{1}{n\sqrt{\ln n}}$ is positive, decreasing

$\int_2^N \frac{1}{x\sqrt{\ln x}}\,dx = \int_{\ln 2}^{\ln N} \frac{1}{\sqrt{u}}\,du = \int_{\ln 2}^{\ln N} u^{-1/2}\,du = 2u^{1/2}\Big|_{\ln 2}^{\ln N} = 2\left(\sqrt{\ln N} - \sqrt{\ln 2}\right)$

$u = \ln x$, $du = \frac{1}{x}dx$

Then $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{N\to\infty} 2\left(\sqrt{\ln N} - \sqrt{\ln 2}\right) = \infty$, integral diverges $\Rightarrow$ sum diverges by integral test

(h) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$

Divergence test: $\lim_{n\to\infty} \frac{(-1)^n n}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{(-1)^n}{1+\frac{2}{n}} = \lim_{n\to\infty} (-1)^n \neq 0$ since $(-1)^n$ diverges

diverges by divergence test

2. Find the Taylor polynomial T_4 of $f(x) = \frac{1}{x}$ about $a = -1$.

$f(x) = x^{-1}$ $\qquad f(-1) = (-1)^{-1} = \frac{1}{-1} = -1$

$f'(x) = -1x^{-2}$ $\qquad f'(-1) = -1\cdot\frac{1}{(-1)^2} = -1$

$f''(x) = (-1)(-2)x^{-3}$ $\qquad f''(-1) = (-1)(-2)\cdot\frac{1}{(-1)^3} = -2 = -2!$

$f'''(x) = (-1)(-2)(-3)x^{-4}$ $\qquad f'''(-1) = (-1)(-2)(-3)\cdot\frac{1}{(-1)^4} = -6 = -3!$

$f^{(4)}(x) = (-1)(-2)(-3)(-4)x^{-5}$ $\qquad f^{(4)}(-1) = (-1)(-2)(-3)(-4)\frac{1}{(-1)^5} = -24 = -4!$

(2 continued)

$$T_4(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$$

$$= -1 + (-1)(x-(-1))^1 + \frac{(-\cancel{2}!)}{\cancel{2}!}(x+1)^2 + \frac{(-\cancel{3}!)}{\cancel{3}!}(x-(-1))^3 + \frac{(-\cancel{4}!)}{\cancel{4}!}(x-(-1))^4$$

$$= \boxed{-1-(x+1)-(x+1)^2-(x+1)^3-(x+1)^4}$$

3. Find $(f^{-1})'(2)$ where $f(x) = x^3 + 3\sin x + 2\cos x$.

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

Two steps ① Find f' ② Find $(f^{-1})(2)$ doesn't matter which one we do first

① Find f': $f(x) = x^3 + 3\sin x + 2\cos x$

$f'(x) = 3x^2 + 3\cos x - 2\sin x$

② Let $x = f^{-1}(2) \Leftrightarrow f(x) = 2$, guess x s.t. $x^3 + 3\sin x + 2\cos x = 2$. $x=0$ works since $f(0) = 0^3 + 3\sin 0 + 2\cos 0 = 0+0+2 = 2$ ✓

Therefore $f^{-1}(2) = 0$.

Plug into formula: $(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{3\cdot 0^2 + 3\cos 0 - 2\sin 0} = \frac{1}{3}$

4. Evaluate the following:

(a) $\int \frac{\tan^{-1}(x)}{1+x^2}dx$

$u = \tan^{-1}x$

$du = \frac{1}{1+x^2}dx$

$$\int \tan^{-1}x \frac{1}{1+x^2}dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\tan^{-1}x)^2}{2} + C$$

(b) $\int \frac{1}{4+x^2}dx = \int \frac{1}{4(1+\frac{x^2}{4})}dx = \frac{1}{4}\int \frac{1}{1+\frac{x^2}{4}}dx$

$u^2 = \frac{x^2}{4} \Rightarrow u = \frac{x}{2}$

$du = \frac{1}{2}dx \Rightarrow 2du = dx$

$$= \frac{1}{4}\int \frac{1}{1+u^2} 2\,du = \frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\arctan u + C = \boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

(c) $\int \frac{1}{\sqrt{5-2x^2}}dx = \int \frac{1}{\sqrt{5}\sqrt{1-\frac{2x^2}{5}}}dx = \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-\frac{2x^2}{5}}}dx$

$u^2 = \frac{2x^2}{5} \Rightarrow u = \sqrt{\frac{2}{5}}x$

$du = \sqrt{\frac{2}{5}}dx \Rightarrow \sqrt{\frac{5}{2}}du = dx$

$$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-u^2}}\sqrt{\frac{5}{2}}du = \frac{1}{\cancel{\sqrt{5}}}\frac{\cancel{\sqrt{5}}}{\sqrt{2}}\int \frac{1}{\sqrt{1-u^2}}du = \frac{1}{\sqrt{2}}\sin^{-1}(u) + C = \boxed{\frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{5}}x\right) + C}$$

(d) $\int \frac{10}{(x-1)(x^2+9)}dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

multiply both sides by $(x-1)(x^2+9)$: $\frac{\cancel{(x-1)(x^2+9)}\,10}{\cancel{(x-1)(x^2+9)}} = \frac{A}{\cancel{x-1}}\cancel{(x-1)}(x^2+9) + \frac{Bx+C}{\cancel{x^2+9}}(x-1)\cancel{(x^2+9)}$

$10 = A(x^2+9) + (Bx+C)(x-1)$

$10 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$

$0x^2 + 0x + 10 = x^2(A+B) + x(-B+C) + 9A - C \Rightarrow$

$A + B = 0$

$-B + C = 0$

$9A - C = 10$

$B = C = -1$

add

$A + C = 0 \rightarrow C = -A = -1$

$9A - C = 10$

$10A = 10$

$A = 1$

Alternate way: $10 = A(x^2+9) + (Bx+C)(x-1)$

Plug in 3 different #'s:

$x=1$: $10 = A(1+9) + (B+C)(1-1) \Rightarrow 10 = 10A \Rightarrow A = 1$

$x=0$: $10 = 1(0+9) + (B\cdot 0 + C)(0-1) \Rightarrow 10 = 9 - C \Rightarrow C = -1$

$x=-1$: $10 = 1(1+9) + (B\cdot(-1) + -1)(-1-1) \Rightarrow 10 = 10 + 2B + 2 \Rightarrow B = -1$

Therefore $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{1}{x-1} + \dfrac{-x-1}{x^2+9}$

$$\Rightarrow \int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{x-1} + \frac{-x-1}{x^2+9}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C$$

$$\int \frac{x}{x^2+9}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+9) + C$$

$u = x^2+9,\ du = 2x\,dx$

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{9\left(\frac{x^2}{9}+1\right)}\,dx = \frac{1}{9}\int \frac{1}{\frac{x^2}{9}+1}\,dx = \frac{1}{9}\int \frac{1}{u^2+1}\,3du = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

$u^2 = \frac{x^2}{9} \Rightarrow u = \frac{x}{3},\ du = \frac{dx}{3}$

Therefore $\displaystyle\int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+9}\,dx = \boxed{\ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C}$

(e) $\displaystyle\int \frac{1}{x^2-16}dx$ $\qquad \dfrac{1}{x^2-16} = \dfrac{1}{(x-4)(x+4)}$, $\dfrac{1}{(x-4)(x+4)} = \dfrac{A}{x-4} + \dfrac{B}{x+4}$

Multiply both sides by $(x-4)(x+4)$: $\quad 1 = A(x+4) + B(x-4)$

$0x + 1 = x(A+B) + 4A - 4B \Rightarrow \begin{cases} A+B=0 \Rightarrow A=-B \\ 4A-4B=1 \end{cases}$ $\quad 4A+4A=1$, $\ 8A=1,\ A=1/8$ $\quad B=-1/8$

Another way: $1 = A(x+4) + B(x-4)$

plug in 2 values for x $\quad x=4:\ 1 = A(4+4) + B(4-4) \Rightarrow A = 1/8$

$x=-4:\ 1 = \frac{1}{8}(4+4) + B(-4-4) \Rightarrow B = -1/8$

Therefore $\displaystyle\int \frac{1}{(x-4)(x+4)}\,dx = \int\left(\frac{1/8}{x-4} - \frac{1/8}{x+4}\right)dx = \frac{1}{8}\int\left(\frac{1}{x-4} + \frac{1}{x+4}\right)dx = \boxed{\frac{1}{8}\left(\ln|x-4| + \ln|x+4|\right) + C}$

(f) $\displaystyle\int \frac{1}{(x+5)^2(x-1)}dx$ $\qquad \dfrac{1}{(x+5)^2(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{(x+5)^2} + \dfrac{C}{x-1}$

Multiply by $(x+5)^2(x-1)$: $\quad 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$

Plug in 3 values for x: $\quad x=1:\ 1 = A(1+5)(1-1) + B(1-1) + C(1+5)^2 \Rightarrow 1 = C(36) \Rightarrow C = 1/36$

$x=-5:\ 1 = A(-5+5)(-5-1) + B(-5-1) + \frac{1}{36}(-5+5)^2 \Rightarrow 1 = -6B \Rightarrow B = -1/6$

$x=0:\ 1 = A(0+5)(0-1) - \frac{1}{6}(0-1) + \frac{1}{36}(0+5)^2 \Rightarrow 1 = -5A + \frac{1}{6} + \frac{25}{36}$

$1 = -5A + \frac{31}{36} \Rightarrow -5A = \frac{5}{36} \Rightarrow A = -1/36$

Therefore $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx = \int\left(\frac{-1/36}{x+5} + \frac{-1/6}{(x+5)^2} + \frac{1/36}{x-1}\right)dx = \boxed{\frac{-1}{36}\ln|x+5| + \frac{1}{6}(x+5)^{-1} + \frac{1}{36}\ln|x-1| + C}$