

Math 31B Worksheet

Week 5

1. Suppose that a_n and b_n are sequences of positive numbers, both of which have a limit of 0 as $n \rightarrow \infty$.

(a) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, then it means that a_n goes to zero faster than b_n .

(b) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, then it means that b_n goes to zero faster than a_n .

Note: If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ is neither 0 nor ∞ , then it means that neither sequence goes to zero faster than the other: a_n and b_n are “*equivalent*” in terms of how fast they go to zero.

2. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(a) Suppose that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} b_n$ diverges then this test is inconclusive.

(b) Suppose that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$.

If $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

If $\sum_{n=1}^{\infty} b_n$ converges then this test is inconclusive.

(c) Suppose that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ is neither 0 nor ∞ . Then what is the relationship between

the convergence/divergence of $\sum_{n=1}^{\infty} a_n$ and the convergence/divergence of $\sum_{n=1}^{\infty} b_n$?

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

3. Consider the series $\sum_{n=3}^{\infty} \frac{4n-2}{n(n-1)(n+1)}$. Call this $\sum_{n=3}^{\infty} a_n$.

(a) What is the fastest growing term in the numerator? n

If you were to multiply out the denominator, what would the fastest-growing term be? (You don't need to actually multiply it out. Just think about what the largest exponent would be if you did.) n^3

(b) Based on your answers to part (a), the terms in the series should go to zero as fast as (i.e., equivalent to) what?

$$\frac{n}{n^3} = \frac{1}{n^2} \quad \sum \frac{1}{n^2}$$

(c) The thing you came up with in part (b)...let's call it b_n . So this defines a new series $\sum_{n=1}^{\infty} b_n$. Does this new, much simpler series converge or diverge?

$$\sum \frac{1}{n^2} \quad \text{p-series, } p=2 \quad \text{converges}$$

(d) Use the Limit Comparison Test to compare the original series $\sum_{n=3}^{\infty} a_n$ to the one from part (c):

First, compute $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.

$$\lim_{n \rightarrow \infty} \frac{\frac{4n}{n(n-1)(n+1)}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{4n}{n(n-1)(n+1)} \cdot \frac{n^2}{1} = \lim_{n \rightarrow \infty} \frac{4n^3}{n^3-n} \cdot \frac{1}{n^3} = \lim_{n \rightarrow \infty} \frac{4}{1-\frac{1}{n^2}} = (4)$$

Based on the value of that limit **and** whether or not $\sum_{n=1}^{\infty} b_n$ converges/diverges,

does $\sum_{n=1}^{\infty} \frac{4n-2}{n(n-1)(n+1)}$ converge or diverge?

$$\sum_{n=1}^{\infty} \frac{4n-2}{n(n-1)(n+1)} \quad \text{converges}$$

4. Consider the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 - n + 5}}$. Call this $\sum_{n=1}^{\infty} a_n$.

(a) In this case, the fastest-growing term in the denominator is n^3 , *but* it's also inside a square root.

So the whole denominator grows as fast as $\frac{\sqrt{n^3} = n^{3/2}}$.

That means the terms in the series, $\frac{n}{\sqrt{n^3 - n + 5}}$, go to zero like $\frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}$.

(b) Repeat the same logic as in the previous problem. First, based on your answer to part (a), what should the series $\sum_{n=1}^{\infty} b_n$ be? Does it converge or diverge?

$$\sum \frac{1}{n^{1/2}} \quad \text{p-series, } p = 1/2 \quad \text{diverges}$$

(c) Use the Limit Comparison Test to compare the original series above to $\sum_{n=1}^{\infty} b_n$:

First, compute $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{n}{\sqrt{n^3 - n + 5}}}{\frac{1}{\sqrt{n}}} &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^3 - n + 5}} \cdot \frac{\sqrt{n}}{1} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^3}{n^3 - n + 5}} \cdot \frac{1}{\sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1 - \frac{1}{n^2} + \frac{5}{n^3}}} = 1 \end{aligned}$$

Based on the value of that limit **and** whether or not $\sum_{n=1}^{\infty} b_n$ converges/diverges,

does the original series above converge or diverge? $\sum \frac{n}{\sqrt{n^3 - n + 5}} \quad \text{diverges}$

5. Consider the series $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^{3/2}}$.

Remember that $\ln(n)$ grows *slower* than any power of n . So the numerator in this series grows slower than $n^{0.000001}$. Therefore, you might think that you can try just

comparing this to the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

(a) Based on what we just said (and on whether or not $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges), formulate

a guess: do you think $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^{3/2}}$ will converge, or diverge?

- (b) Try using the Limit Comparison Test to compare $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^{3/2}}$ to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. What happens, and why?

$$\lim_{n \rightarrow \infty} \frac{\frac{\ln(n+1)}{n^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \ln(n+1) = \infty \quad \text{but } \sum \frac{1}{n^{3/2}} \text{ converges} \Rightarrow \text{inconclusive}$$

- (c) Okay, so $\ln(n)$ grows slowly, but it's not completely insignificant in this problem. So let's try comparing our series to something that goes to zero a little slower than $\frac{1}{n^{3/2}}$, like, say, $\frac{1}{n}$. That is, try using the Limit Comparison Test to compare $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^{3/2}}$ to $\sum_{n=1}^{\infty} \frac{1}{n}$. What is the problem this time?

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{\ln(n+1)}{n^{3/2}}}{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n^{1/2}} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{2} n^{-1/2}} = \lim_{n \rightarrow \infty} \frac{2n^{1/2}}{n+1} \stackrel{\frac{1/2}}{\frac{1/2}} \\ &= \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n} + \frac{1}{\sqrt{n}}} = 0 \end{aligned}$$

but $\sum \frac{1}{n}$ diverges \Rightarrow inconclusive

- (d) Since neither attempt in (b) or (c) worked, what might you try now? Remember, you're trying to prove that the series converges, so you need to compare to another series that does the same. But also you need to use an exponent in the denominator that's at least a little smaller than $\frac{3}{2}$.

Come up with a b_n that satisfies those requirements, and try the Limit Comparison Test on it. You should be able to prove your guess from part (a).

We tried $\sum \frac{1}{n^{1.5}}$ and $\sum \frac{1}{n^1}$, try $\sum \frac{1}{n^p}$ for some $k < p < 1.5$.

I will try $p = 1.25$, $\sum \frac{1}{n^{1.25}}$ converges since $p = 1.25 > 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{\ln(n+1)}{n^{1.5}}}{\frac{1}{n^{1.25}}} &= \lim_{n \rightarrow \infty} \frac{n^{1.25} \ln(n+1)}{n^{1.5}} = \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n^{0.25}} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{0.25}{n^{0.75}}} \\ &= \lim_{n \rightarrow \infty} \frac{n^{0.75}}{0.25(n+1)} \stackrel{\frac{1/0.75}{1/0.75}}{=} \lim_{n \rightarrow \infty} \frac{1}{0.25(n^{0.25} + \frac{1}{n^{0.75}})} = 0 \end{aligned}$$

$\Rightarrow \sum \frac{\ln(n+1)}{n^{3/2}}$ converges by 2(b)

