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Math 31B Worksheet
Week 6

1. Use the Ratio Test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{10^n}{n!}$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} \right| = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \lim_{n\to\infty} \frac{10^{n+1}}{10^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{10^n}\, 10^1}{\cancel{10^n}} \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty} \frac{10}{n+1} = 0 < 1$$

converges by ratio test

2. Use the Ratio Test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(2(n+1))!}}{\frac{n!}{(2n)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{\overbrace{n+1}^{\text{degree 1}}}{\underbrace{(2n+2)(2n+1)}_{\text{degree 2}}} = 0$$

3. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$.

(a) Use the Ratio Test to determine whether or not this series converges.

$$L = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n (n+1)} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \underbrace{\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n}_{\text{need to make sure this is not } \infty} \underbrace{\lim_{n\to\infty} \frac{1}{n+1}}_{\to 0}$$

indeterminant $\frac{0}{0}$ since $\frac{n}{n+1} \to 1$ as $n\to\infty$

$$\text{let } K = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \Rightarrow \ln K = \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln(n) - \ln(n+1)}{\frac{1}{n}}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n+1}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{\cancel{n}+1-\cancel{n}}{n(n+1)}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n(n+1)} \cdot \frac{-n^2}{1} = \lim_{n\to\infty} \frac{-n^2}{n^2+n} \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{-1}{1+\frac{1}{n}} = -1 \Rightarrow \ln K = -1 \Rightarrow K = e^{-1} \Rightarrow L = e^{-1} \cdot 0 = 0 < 1$$

converges by ratio test

(b) Now use the Root Test to do the same. Which one is easier in this case?

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1,$$ converges by root test

Root test is much easier for this problem

4. List the following functions in order from growing the slowest to growing the fastest.

$$n \quad n! \quad 2^n \quad n^2 \quad \sqrt{n} \quad n^3 \quad n^{1000} \quad n^{0.001} \quad n^n \quad \ln(n) \quad \ln(\ln(n)) \quad 10^n \quad e^n$$

$\ln(\ln n) \ll \underline{\ln n} \ll \underline{n^{0.001}} \ll \underline{\sqrt{n}} \ll \underline{n} \ll \underline{n^2} \ll \underline{n^3} \ll \underline{n^{1000}} \ll \underline{2^n} \ll \underline{e^n} \ll \underline{10^n} \ll \underline{n!} \ll \underline{n^n}$

For each of the following series, identify the *dominant term* (i.e., fastest-growing term) in the numerator and the denominator of the terms of the series. Use this to come up with a guess about whether the series converges or diverges.

→ no right or wrong answer, just guessing

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100} + 6n^5 + \sqrt{n}}{1.01^n - ln(n)}$ $\displaystyle\frac{n^{100}}{1.01^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n + 10^n}{(2n)!}$ $\displaystyle\frac{n^n}{(2n)!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n^4 + 1)}{n^{1.02}}$ $\displaystyle\frac{\ln(n^4+1)}{n^{1.02}}$

For each of the following series, choose an appropriate convergence test, and use it to show that the series converges or diverges.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}}{1.01^n}$

Ratio: $L = \lim\limits_{n\to\infty} \left| \dfrac{\frac{(n+1)^{100}}{1.01^{n+1}}}{\frac{n^{100}}{1.01^n}} \right| = \lim\limits_{n\to\infty} \dfrac{(n+1)^{100}}{1.01^{n+1}} \cdot \dfrac{1.01^n}{n^{100}} = \lim\limits_{n\to\infty} \dfrac{(n+1)^{100}}{n^{100}} \cdot \dfrac{1.01^n}{1.01^{n+1}}$

$= \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^{100} \cdot \dfrac{1}{1.01} = \lim\limits_{n\to\infty} \cancel{\left(1+\frac{1}{n}\right)^{100}} \overset{\to 1}{} \dfrac{1}{1.01} = \dfrac{1}{1.01} < 1$

Converges by ratio test

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$

Ratio Test: $L = \lim\limits_{n\to\infty} \dfrac{(n+1)^{n+1}}{(2(n+1))!} \cdot \dfrac{(2n)!}{n^n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^{n+1}}{n^n} \cdot \dfrac{(2n)!}{(2n+2)!}$

$= \lim\limits_{n\to\infty} \dfrac{(n+1)^n(n+1)}{n^n} \cdot \dfrac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = \lim\limits_{n\to\infty} \underbrace{\left(\dfrac{n+1}{n}\right)^n}_{\text{need to find this limit}} \underbrace{\dfrac{n+1}{(2n+2)(2n+1)}}_{\to 0}$

Let $K = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^n \Rightarrow \ln K = \lim\limits_{n\to\infty} n \ln\left(\dfrac{n+1}{n}\right) = \lim\limits_{n\to\infty} \dfrac{\ln\left(\frac{n+1}{n}\right)}{\frac{1}{n}} = \dfrac{0}{0}$ since $\dfrac{n+1}{n} \to 1$ as $n\to\infty$

$= \lim\limits_{n\to\infty} \dfrac{\ln(n+1) - \ln n}{\frac{1}{n}} \overset{LH}{=} \lim\limits_{n\to\infty} \dfrac{\frac{1}{n+1} - \frac{1}{n}}{\frac{-1}{n^2}} = \lim\limits_{n\to\infty} \dfrac{\cancel{n} - (\cancel{n}+1)}{n(n+1)} \cdot \dfrac{-n^2}{1}$

$= \lim\limits_{n\to\infty} \dfrac{n^2}{n^2+n} \dfrac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1$

$\ln K = 1 \Rightarrow K = e^1 \Rightarrow L = K \cdot 0 = e \cdot 0 = 0 < 1$, Converges by ratio test

* Similar to Week 5 worksheet

10. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n^4+1)}{n^{1.02}}$

Limit Comparison: Try comparing with $\sum \frac{1}{n^{1.02}}$, which converges by p-series

$\lim\limits_{n\to\infty} \dfrac{\frac{\ln(n^4+1)}{n^{1.02}}}{\frac{1}{n^{1.02}}} = \lim\limits_{n\to\infty} \ln(n^4+1) = \infty$, inconclusive

Try comparing with $\sum \frac{n^{0.02}}{n^{1.02}} = \sum \frac{1}{n}$, which diverges by p-series

$\lim\limits_{n\to\infty} \dfrac{\frac{\ln(n^4+1)}{n^{1.02}}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{\ln(n^4+1)}{n^{0.02}} \overset{LH}{=} \lim\limits_{n\to\infty} \dfrac{\frac{1}{n^4+1} \cdot 4n^3}{0.02n^{-0.98}} = \dfrac{1}{0.02} \lim\limits_{n\to\infty} \dfrac{4n^{3.98}}{n^4+1} \dfrac{\frac{1}{n^{3.98}}}{\frac{1}{n^{3.98}}} = \dfrac{1}{0.02} \lim\limits_{n\to\infty} \dfrac{4}{n^{0.02} + \frac{1}{n^{3.98}}} = 0$

Need to compare w/ $\sum \frac{1}{n^p}$ where $1 < p < 1.02$, Try comparing w/ $\sum \frac{1}{n^{1.01}}$, which converges by p-series

$\lim\limits_{n\to\infty} \dfrac{\frac{\ln(n^4+1)}{n^{1.02}}}{\frac{1}{n^{1.01}}} = \lim\limits_{n\to\infty} \dfrac{\ln(n^4+1)}{n^{0.01}} \overset{LH}{=} \lim\limits_{n\to\infty} \dfrac{\frac{1}{n^4+1} \cdot 4n^3}{0.01\, n^{-0.99}} = \dfrac{1}{0.01} \lim\limits_{n\to\infty} \dfrac{4n^{3.99}}{n^4+1} \dfrac{\frac{1}{n^{3.99}}}{\frac{1}{n^{3.99}}} = \dfrac{1}{0.01} \lim\limits_{n\to\infty} \dfrac{4}{n^{0.01} + \frac{1}{n^{3.99}}} = 0$

This means $\sum \frac{1}{n^{1.01}}$ is "bigger" than $\sum \frac{\ln(n^4+1)}{n^{1.02}}$, and since the "bigger" one converges, the smaller must converge

$\Rightarrow$ Converges by limit comparison

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\big(\ln(n)\big)^2}$

Integral Test : $\frac{1}{n(\ln n)^2}$ is decreasing and positive (can check decreasing by showing derivative <0)

$\int_2^N \frac{1}{x(\ln x)^2}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}dx$

when $x=N$, $u=\ln N$
$x=2$, $u=\ln 2$

$\int_2^N \underbrace{\frac{1}{(\ln x)^2}}_{u} \cdot \underbrace{\frac{1}{x}dx}_{du} = \int_{\ln 2}^{\ln N} \frac{1}{u^2}\,du = \int_{\ln 2}^{\ln N} u^{-2}\,du$

$= \frac{u^{-1}}{-1}\Big|_{\ln 2}^{\ln N} = -\frac{1}{u}\Big|_{\ln 2}^{\ln N} = -\left(\frac{1}{\ln N} - \frac{1}{\ln 2}\right)$

Take limit as $N \to \infty$: $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{N\to\infty} \int_2^N \frac{1}{x(\ln x)^2}\,dx = \lim_{N\to\infty} -\left(\frac{1}{\ln N} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$, integral converges

converges by integral test

12. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\ln(n)}$

Alternating series test : $b_n = \frac{1}{\ln n}$

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$ ✓
- b_n is decreasing since $\left(\frac{1}{\ln x}\right)' = \left[(\ln x)^{-1}\right]' = -1(\ln x)^{-2} \cdot \frac{1}{x} = \frac{-1}{x(\ln x)^2} < 0$

converges by alternating series test

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{2+3n-6n^2}$

Divergence test, $\lim_{n\to\infty} \frac{n^2 + n^{1/2}}{2+3n-6n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n^{3/2}}}{\frac{2}{n^2}+\frac{3}{n}-6} = \frac{-1}{6} \neq 0$

diverges by divergence test