Solutions written by Victoria Kala
vtkala@math.ucla.edu

Math 31B Worksheet
Week 7

1. For this whole problem, we will use the function $f(x) = \dfrac{1}{1-x}$.

   (a) Write down the fifth-degree Taylor polynomial for $f(x)$, centered at $x = 0$. (this is value for a)

$f(x) = (1-x)^{-1}$ | $f(0)=1$
$f'(x) = -1(1-x)^{-2}(-1) = 1(1-x)^{-2}$ | $f'(0)=1$
$f''(x) = 1(-2)(1-x)^{-3}(-1) = 1\cdot 2(1-x)^{-3}$ | $f''(0) = 2!$
$f'''(x) = 1\cdot 2(-3)(1-x)^{-4}(-1) = 1\cdot 2\cdot 3(1-x)^{-4}$ | $f'''(0) = 3!$
$f^{(4)}(x) = 1\cdot 2\cdot 3(-4)(1-x)^{-5}(-1) = 1\cdot 2\cdot 3\cdot 4(1-x)^{-5}$ | $f^{(4)}(0) = 4!$
$f^{(5)}(x) = 1\cdot 2\cdot 3\cdot 4(-5)(1-x)^{-6}(-1) = 1\cdot 2\cdot 3\cdot 4\cdot 5(1-x)^{-6}$ | $f^{(5)}(0) = 5!$

$T_5(x) = 1 + \frac{1}{1!}(x-0) + \frac{2!}{2!}(x-0)^2 + \frac{3!}{3!}(x-0)^3 + \frac{4!}{4!}(x-0)^4 + \frac{5!}{5!}(x-0)^5$
$= \boxed{1 + x + x^2 + x^3 + x^4 + x^5}$

   (b) Do you notice a pattern in the terms of the Taylor polynomial you just wrote? Can you come up with a general formula for the n^{th}-degree Taylor polynomial $T_n(x)$?

$T_n(x) = 1 + x + x^2 + \dots + x^n$

   (c) What is $f(-0.5)$? Compute $T_1(-0.5)$, $T_2(-0.5)$, $T_3(-0.5)$, $T_4(-0.5)$, and $T_5(-0.5)$. For each one of these, also compute the error $|f(-0.5) - T_n(-0.5)|$. What do you notice about the error, as you add more and more terms to the Taylor polynomial?

$f(-0.5) = \frac{1}{1+0.5} = \frac{2}{3}$

$T_1(x) = 1+x$	$T_1(-0.5) = 1-0.5 = 0.5$	$\lvert T_1(-0.5) - f(-0.5)\rvert = 0.1666\ldots$
$T_2(x) = 1+x+x^2$	$T_2(-0.5) = 1-0.5+(0.5)^2 = 0.75$	$\lvert T_2(-0.5) - f(-0.5)\rvert = 0.0833\ldots$
$T_3(x) = 1+x+x^2+x^3$	$T_3(-0.5) = 1-0.5+(0.5)^2-(0.5)^3 = 0.625$	$\lvert T_3(-0.5) - f(-0.5)\rvert = 0.04166\ldots$
$T_4(x) = 1+x+x^2+x^3+x^4$	$T_4(-0.5) = 1-0.5+(0.5)^2-(0.5)^3+(0.5)^4 = 0.6875$	$\lvert T_4(-0.5) - f(-0.5)\rvert = 0.020833\ldots$
$T_5(x) = 1+x+x^2+x^3+x^4+x^5$	$T_5(-0.5) = 1-0.5+(0.5)^2-(0.5)^3+(0.5)^4-(0.5)^5 = 0.65625$	$\lvert T_5(-0.5) - f(-0.5)\rvert = 0.0104166\ldots$

* error decreases as n increases (cool observation: error of T_n is ½ error of T_{n-1})

   (d) What is $f(-1.5)$? Compute $T_1(-1.5)$, $T_2(-1.5)$, $T_3(-1.5)$, $T_4(-1.5)$, and $T_5(-1.5)$. For each one of these, also compute the error $|f(-1.5) - T_n(-1.5)|$. This time, what happens to the error as you add more terms to the Taylor polynomial?

$T_1, \dots, T_5$ same as above

$f(-1.5) = \frac{1}{1+1.5} = \frac{2}{5}$

$T_1(-1.5) = 1-1.5 = -0.5$	$\lvert T_1(-1.5) - f(-1.5)\rvert = 0.9$
$T_2(-1.5) = 1-1.5+(1.5)^2 = 1.75$	$\lvert T_2(-1.5) - f(-1.5)\rvert = 1.35$
$T_3(-1.5) = 1-1.5+(1.5)^2-(1.5)^3 = -1.625$	$\lvert T_3(-1.5) - f(-1.5)\rvert = 2.025$
$T_4(-1.5) = 1-1.5+(1.5)^2-(1.5)^3+(1.5)^4 = 3.4375$	$\lvert T_4(-1.5) - f(-1.5)\rvert = 3.0375$
$T_5(-1.5) = 1-1.5+(1.5)^2-(1.5)^3+(1.5)^4-(1.5)^5 = -4.15625$	$\lvert T_5(-1.5) - f(-1.5)\rvert = 4.55625$

* errors increase as n increases

Summary: Taylor polynomials are good approximations to $f(x)$ as long as we evaluate "close" to the center we expanded about
-0.5 was close to $a=0$, so $T_n(-0.5)$ is a good approximation to $f(-0.5)$
-1.5 is not close to $a=0$, so $T_n(-1.5)$ is not a good approximation to $f(1.5)$

2. If f is a function, and g is its inverse, then

$$\text{if } f(x) = y, \text{ then } g(\underline{\ y\ }) = \underline{\ x\ }.$$

So, for example,

$$\text{if } f(2) = 7, \text{ then } g(\underline{\ 7\ }) = \underline{\ 2\ }, \text{ and}$$
$$\text{if } f(-1) = 3, \text{ then } g(\underline{\ 3\ }) = \underline{\ -1\ }.$$

3. Suppose f is a function, and g is its inverse. The Inverse Function Theorem says the following: if f is differentiable and its derivative is not zero, then g is also differentiable, and its derivative is (fill in the missing part)

$$g'(x) = \frac{1}{f'(g(x))}$$

4. Suppose $f(x) = x^5 + 3x + 1$. It turns out that f is invertible, but its inverse function (which we'll call g) cannot be expressed by any formula! (I.e., g is not an *elementary function.*)

(a) What is $g(5)$? *(Hint: See problem 2 above. Can you come up with an x for which $f(x) = 5$?)*

$x^5+3x+1=5 \leftarrow$ guess x so that this is true

$x=1$ works, so $f(1)=5 \Rightarrow g(5)=1$

(b) Use the Inverse Function Theorem to compute $g'(5)$.

$f(x)=x^5+3x+1$

$f'(x)=5x^3+3$

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(1)} = \frac{1}{5\cdot 1^3+3} = \frac{1}{8}$$

5. Suppose $f(x) = \sin(x) + 2x$. Once again, this is a function that is invertible, but its inverse cannot be expressed by any formula. Let's again let g refer to the inverse of f, that is, $g = f^{-1}$.

(a) What is $g(0)$?

$\sin x+2x=0$, $x=0$ makes this happen

$f(0)=0 \Rightarrow g(0)=0$

(b) Use the Inverse Function Theorem to compute $g'(0)$.

$$g'(0) = \frac{1}{f'(g(0))} = \frac{1}{f'(0)} = \frac{1}{\cos 0+2} = \frac{1}{3}$$

$f(x)=\sin x+2x$

$f'(x)=\cos x+2$