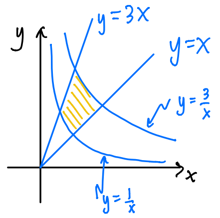
Solutions to Practice Problems II

Written by Victoria Kala

1. Evaluate $\iint_R xy\,dA$ where R is the region in the first quadrant bounded by the lines $y = x$, $y = 3x$ and the hyperbolas $xy = 1, xy = 3$. Use the transformation $x = u/v$, $y = v$.



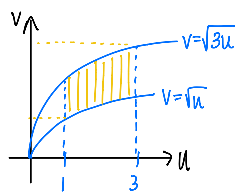
$xy = 1 \Rightarrow y = \frac{1}{x}$ at $x = 1, y = 1$

$xy = 3 \Rightarrow y = \frac{3}{x}$ at $x = 1, y = 3$

What does this look like in the uv-plane?
Plug in $x = \frac{u}{v}$ and $y = v$ into the four equations:

$y = x \Rightarrow v = \frac{u}{v} \Rightarrow u = v^2 \Rightarrow v = \sqrt{u}$ $\qquad$ $xy = 1 \Rightarrow \frac{u}{v}\cdot v = 1 \Rightarrow u = 1$

$y = 3x \Rightarrow v = \frac{3u}{v} \Rightarrow u = \frac{v^2}{3} \Rightarrow v = \sqrt{3u}$ $\qquad$ $xy = 3 \Rightarrow \frac{u}{v}\cdot v = 3 \Rightarrow u = 3$



Bounds for new domain: $1 \le u \le 3$
$\sqrt{u} \le v \le \sqrt{3u}$

Jacobian: $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}\left(\frac{u}{v}\right) & \frac{\partial}{\partial v}\left(\frac{u}{v}\right) \\ \frac{\partial}{\partial u}(v) & \frac{\partial}{\partial v}(v) \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$

Then $\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\cdot v \cdot \underbrace{\frac{1}{v}}_{\text{Jacobian}}\,dv\,du = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\,dv\,du = \int_1^3 u \ln v \Big|_{v=\sqrt{u}}^{\sqrt{3u}} du$

$= \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du$, but $\ln\sqrt{3u} - \ln\sqrt{u} = \frac{1}{2}\ln(3u) - \frac{1}{2}\ln u = \frac{1}{2}(\ln(3u) - \ln u)$
$= \frac{1}{2}\ln\left(\frac{3u}{u}\right) = \frac{1}{2}\ln 3$

$= \int_1^3 u\,\frac{1}{2}\ln 3\,du = \frac{1}{2}\ln 3 \cdot \frac{u^2}{2}\Big|_1^3 = \frac{1}{2}\ln 3 \cdot \frac{1}{2}(9-1) = 2\ln 3$

2. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F}(x,y) = e^x \cos y\,\mathbf{i} + e^x \sin y\,\mathbf{j}$

$\frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y \longleftarrow$ not the same $\longrightarrow \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ $\qquad$ Not conservative

(b) $\mathbf{F}(x,y) = (\ln y + 2xy^3)\mathbf{i} + \left(3x^2y^2 + \frac{x}{y}\right)\mathbf{j}$

$\frac{\partial}{\partial y}(\ln y + 2xy^3) = \frac{1}{y} + 6xy^2 \longleftarrow$ same! $\longrightarrow \frac{\partial}{\partial x}\left(3x^2y^2 + \frac{x}{y}\right) = 6xy^2 + \frac{1}{y}$ $\qquad$ conservative

Now set $\nabla f = \vec{F} \Rightarrow \langle f_x, f_y \rangle = \langle \ln y + 2xy^3, 3x^2y^2 + \frac{x}{y} \rangle$

$f_x = \ln y + 2xy^3 \Rightarrow f = \int(\ln y + 2xy^3)\,dx = x\ln y + x^2y^3 + g(y) \Rightarrow \boxed{f = x\ln y + x^2y^3 + C}$

$f_y = 3x^2y^2 + \frac{x}{y} \Rightarrow f = \int\left(3x^2y^2 + \frac{x}{y}\right)dy = x^2y^3 + x\ln y + h(x)$

3. Evaluate the line integral $\int_C xyz\,ds$ where C is the curve parametrized by $x = 2\sin t, y = t, z = -2\cos t, 0 \le t \le \pi$.

Given $\vec{r}(t) = \langle 2\sin t, t, -2\cos t \rangle, \quad 0 \le t \le \pi$

$\vec{r}'(t) = \langle 2\cos t, 1, 2\sin t \rangle \qquad \|\vec{r}'(t)\| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t} = \sqrt{5}$ since $\sin^2 t + \cos^2 t = 1$

Then $\int_C xyz\,ds = \int_0^\pi (2\sin t)t(-2\cos t)\sqrt{5}\,dt = -2\sqrt{5}\int_0^\pi t \cdot 2\sin t\cos t\,dt = -2\sqrt{5}\int_0^\pi t\sin(2t)\,dt$

since $\sin(2t) = 2\sin t\cos t$

Parts: $u = t \quad dv = \sin 2t\,dt$; $du = 1\,dt \quad v = -\frac{1}{2}\cos 2t$

$= -2\sqrt{5}\left[-\frac{1}{2}t\cos 2t \Big|_0^\pi + \int_0^\pi \frac{1}{2}\cos 2t\,dt\right] = -2\sqrt{5}\left[-\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t\right]_0^\pi$

$= -2\sqrt{5}\left[-\frac{1}{2}\pi\cos 2\pi + \cancel{\frac{1}{4}\sin 2\pi}^{\,0} - (0+0)\right] = \sqrt{5}\pi$

4. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is given by the vector function $\mathbf{r}(t) = t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}, 0 \le t \le \pi$ and $\mathbf{F} = z\mathbf{i} + y\mathbf{j} - x\mathbf{k}$.

Given $\vec{r}(t) = \langle \overset{x}{t}, \overset{y}{\sin t}, \overset{z}{\cos t} \rangle, \quad 0 \le t \le \pi$

$\vec{r}'(t) = \langle 1, \cos t, -\sin t \rangle$

Then $\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_0^\pi \langle \cos t, \sin t, -t \rangle \cdot \langle 1, \cos t, -\sin t \rangle\,dt$

$= \int_0^\pi (\cos t + \cos t\sin t + t\sin t)\,dt$

$\int \cos t\,dt = \sin t \qquad \int \cos t\sin t\,dt = \frac{1}{2}\int \sin 2t\,dt = -\frac{1}{4}\cos 2t \qquad \int t\sin t\,dt = -t\cos t + \int \cos t\,dt$

parts $u = t \quad dv = \sin t\,dt$; $du = dt \quad v = -\cos t$ $\qquad = -t\cos t + \sin t$

$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \left(\sin t - \frac{1}{4}\cos 2t - t\cos t + \sin t\right)\Big|_0^\pi = 0 - \cancel{\frac{1}{4}\cos 2\pi} - \pi(-1) + 0 - \left(0 - \cancel{\frac{1}{4}} - 0 + 0\right) = \pi$

5. Let $\mathbf{F}(x, y, z) = y^2\cos z\mathbf{i} + 2xy\cos z\mathbf{j} - xy^2\sin z\mathbf{k}$, C be the curve parametrized by $\mathbf{r}(t) = t^2\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, 0 \le t \le \pi$.

(a) Show that $\mathbf{F}$ is conservative.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos z & 2xy\cos z & -xy^2\sin z \end{vmatrix} = \hat{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2xy\cos z & -xy^2\sin z \end{vmatrix} - \hat{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ y^2\cos z & -xy^2\sin z \end{vmatrix} + \hat{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ y^2\cos z & 2xy\cos z \end{vmatrix}$$

$= \langle \frac{\partial}{\partial y}(-xy^2\sin z) - \frac{\partial}{\partial z}(2xy\cos z), -\frac{\partial}{\partial x}(-xy^2\sin z) + \frac{\partial}{\partial z}(y^2\cos z), \frac{\partial}{\partial x}(2xy\cos z) - \frac{\partial}{\partial y}(y^2\cos z) \rangle$

$= \langle -2xy\sin z + 2xy\sin z, y^2\sin z - y^2\sin z, 2y\cos z - 2y\cos z \rangle = \vec{0}$ ✓

(b) Find a function f such that $\mathbf{F} = \nabla f$.

$\langle f_x, f_y, f_z \rangle = \langle y^2\cos z, 2xy\cos z, -xy^2\sin z \rangle$

$f_x = y^2\cos z \qquad f = \int y^2\cos z\,dx = xy^2\cos z + g(y,z)$

$f_y = 2xy\cos z \quad \longrightarrow \quad f = \int 2xy\cos z\,dy = xy^2\cos z + h(x,z)$

$f_z = -xy^2\sin z \qquad f = \int -xy^2\sin z\,dz = xy^2\cos z + k(x,y)$

$\boxed{f = xy^2\cos z + C}$

(c) Use (b) to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

Fundamental Theorem of Line Integrals: $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

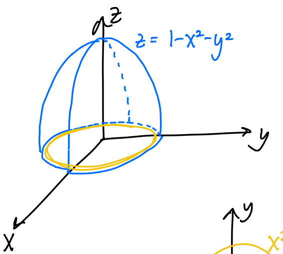
Here $\vec{r}(t) = \langle t^2, \sin t, t \rangle, \quad 0 \le t \le \pi$

$\vec{r}(\pi) = \langle \pi^2, 0, \pi \rangle$

$\vec{r}(0) = \langle 0, 0, 0 \rangle$

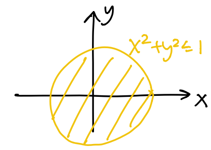
So then $\int_C \vec{F} \cdot d\vec{r} = xy^2\cos z\Big|_{(\pi^2,0,\pi)} - xy^2\cos z\Big|_{(0,0,0)} = 0$

6. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



We actually have 2 surfaces: ① paraboloid (S_1)
② $z=0$ plane intersects paraboloid (S_2)

$$\iint_S \vec F\cdot d\vec S = \iint_{S_1} \vec F\cdot d\vec S + \iint_{S_2} \vec F\cdot d\vec S$$



S_1: parametrize $\vec r(x,y,z) = \langle x,y,z\rangle$
$\vec r(x,y) = \langle x, y, 1-x^2-y^2\rangle$

Bounds for x,y is the disk $x^2+y^2\le 1$ (where $z=0$ plane intersects paraboloid)

Change to polar: $\vec r(r,\theta) = \langle \overset{x}{r\cos\theta}, \overset{y}{r\sin\theta}, \overset{z}{1-r^2}\rangle$
$0\le r\le 1, \quad 0\le\theta\le 2\pi$

$\vec r_r = \langle \cos\theta, \sin\theta, -2r\rangle, \quad \vec r_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec r_r\times\vec r_\theta = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = \hat\imath\begin{vmatrix}\sin\theta & -2r \\ r\cos\theta & 0\end{vmatrix} - \hat\jmath\begin{vmatrix}\cos\theta & -2r \\ -r\sin\theta & 0\end{vmatrix} + \hat k\begin{vmatrix}\cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta\end{vmatrix}$$

$= \langle 2r^2\cos\theta, 2r^2\sin\theta, r\cos^2\theta + r\sin^2\theta\rangle = \langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle$ (turns out this is the outward parametrization)

$\vec F = \langle y, x, z\rangle$

$$\iint_{S_1} \vec F\cdot d\vec r = \iint_{S} \vec F(\vec r(t))\cdot\vec r'(t)\,dt = \int_0^{2\pi}\int_0^1 \langle r\sin\theta, r\cos\theta, 1-r^2\rangle\cdot\langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2r^3\sin\theta\cos\theta + 2r^3\sin\theta\cos\theta + r - r^3)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (2r^3\sin 2\theta + r - r^3)\,dr\,d\theta$$ since $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \int_0^{2\pi}\left(\frac{r^4}{2}\sin 2\theta + \frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_{r=0}^{1} d\theta = \int_0^{2\pi}\left(\frac12\sin 2\theta + \frac12 - \frac14\right)d\theta = \int_0^{2\pi}\left(\frac12\sin 2\theta + \frac14\right)d\theta$$

$$= -\frac14\cos 2\theta + \frac14\theta\Big|_0^{2\pi} = \frac{\pi}{2}$$

S_2: parametrize $\vec r(x,y,z) = \langle x,y,z\rangle$
$\vec r(x,y) = \langle x,y,0\rangle$ since $z=0$

Domain still $x^2+y^2\le 1 \Rightarrow$ polar. $\vec r(r,\theta) = \langle \underset{x}{r\cos\theta}, \underset{y}{r\sin\theta}, \underset{z}{0}\rangle \quad 0\le r\le 1, \ 0\le\theta\le 2\pi$

$\vec r_r = \langle\cos\theta, \sin\theta, 0\rangle$
$\vec r_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec r_r\times\vec r_\theta = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = \hat k\begin{vmatrix}\cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta\end{vmatrix} = \langle 0,0,r\rangle$$ (turns out this is inward parametrization)

Want $\vec r_\theta\times\vec r_r = \langle 0,0,-r\rangle$

$$\iint_{S_2}\vec F\cdot d\vec r = \int_0^{2\pi}\int_0^1 \langle r\sin\theta, r\cos\theta, 0\rangle\cdot\langle 0,0,-r\rangle\,dr\,d\theta = \int_0^{2\pi}\int_0^1 0\,dr\,d\theta = 0 \rightarrow \text{sweet!}$$

$\vec F = \langle y,x,z\rangle$

Therefore $\iint_S \vec F\cdot d\vec S = \frac{\pi}{2} + 0 = \frac{\pi}{2}$