Math 32B Week 5 Worksheet

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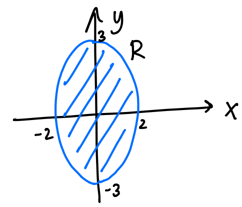
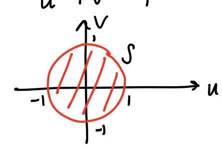
1. Evaluate $\iint_R x^2 dA$ where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$. Use the transformation $x = 2u, y = 3v$.

Step 0 Sketch old region $\frac{x^2}{4} + \frac{y^2}{9} = 1$



Step 1 Sketch new region.

$x = 2u$ and $y = 3v \Rightarrow 9(2u)^2 + 4(3v)^2 = 36 \Rightarrow 36u^2 + 36v^2 = 36 \Rightarrow u^2 + v^2 = 1$



Step 2 Find the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Step 3 Plug in

$$\iint_R x^2 \, dA = \iint_S (2u)^2 \, |6| \, dA = 24 \iint u^2 dA$$

Change to polar. $u = r\cos\theta$ $\quad 0 \le r \le 1$ $\quad dA = r\,dr\,d\theta$
$v = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$$24 \int_0^{2\pi} \int_0^1 r^2 \cos^2\theta \, r \, dr\,d\theta = 24 \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^1 r^3 dr = 24 \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) d\theta \cdot \frac{1}{4} r^4 \Big|_0^1$$

$$= 3\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = 6\pi$$

2. Determine whether or not the following vector fields are conservative:

(a) $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$

$$\frac{\partial(x-y)}{\partial y} = -1, \quad \frac{\partial(x-2)}{\partial x} = 1 \quad \Big\} \text{ not the same, not conservative}$$

(b) $\mathbf{F}(x, y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$

$$\frac{\partial}{\partial y}(3+2xy) = 2x, \quad \frac{\partial}{\partial x}(x^2-3y^2) = 2x \quad \Big\} \text{ same, conservative}$$

3. Let $\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$.

(a) Show that $\mathbf{F}$ is conservative.

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & 2xy+e^{3z} & 3ye^{3z} \end{vmatrix} = \hat{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2xy+e^{3z} & 3ye^{3z} \end{vmatrix} - \hat{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ y^2 & 3ye^{3z} \end{vmatrix} + \hat{k}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ y^2 & 3ye^{3z} \end{vmatrix}$$

$$= \hat{i}\Big(\underbrace{\frac{\partial}{\partial y}3ye^{3z} - \frac{\partial}{\partial z}(2xy+e^{3z})}_{3e^{3z} - 3e^{3z} = 0}\Big) - \hat{j}\Big(\frac{\partial}{\partial x}(3ye^{3z}) - \frac{\partial}{\partial z}(y^2)\Big) + \hat{k}\Big(\frac{\partial}{\partial x}(3ye^{3z}) - \frac{\partial}{\partial z}(y^2)\Big)$$

0 0 0

$$= \vec{0} \Rightarrow \text{conservative}$$

(b) Find a function f such that $\nabla f = F$.

$$\nabla f = \vec{F} \Rightarrow \langle f_x, f_y, f_z \rangle = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$$

$$f_x = y^2 \Rightarrow f = \int y^2\,dx = xy^2 + g(y,z)$$

$$f_y = 2xy + e^{3z} \Rightarrow f = \int (2xy + e^{3z})\,dy = xy^2 + ye^{3z} + h(x,z)$$

$$f_z = 3ye^{3z} \Rightarrow f = \int 3ye^{3z}\,dz = ye^{3z} + m(x,y)$$

$$\boxed{f = xy^2 + ye^{3z} + C}$$