Math 32B Week 6 Worksheet

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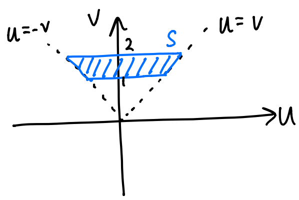
1. Evaluate the integral $\iint_R e^{(x+y)/(x-y)}dA$ where R is the trapezoidal region with vertices $(1,0),(2,0),(0,-2),(0,-1)$. Use the transformation $x=\frac{1}{2}(u+v), y=\frac{1}{2}(u-v)$. *Hint:* The region of integration S in the uv-plane is $\{(u,v): 1\le v\le 2, -v\le u\le v\}$.



Already given the bounds in the uv plane.

Jacobian: $J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix} = \frac{1}{2}\left(-\frac{1}{2}\right)-\frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$

$$\iint_S f(x(u,v),y(u,v))\,|J|\,du\,dv = \int_1^2\int_{-v}^{v} e^{\left(\frac{1}{2}(u+v)+\frac{1}{2}(u-v)\right)/\left(\frac{1}{2}(u+v)-\frac{1}{2}(u-v)\right)}\left|-\frac{1}{2}\right|du\,dv = \frac{1}{2}\int_1^2\int_{-v}^{v} e^{u/v}\,du\,dv$$

let $w=\frac{u}{v}$, $dw=\frac{1}{v}du$, $v\,dw=du$

when $u=-v$, $w=-1$; $u=v$, $w=1$

$$=\frac{1}{2}\int_1^2\int_{-1}^{1} e^{w}v\,dw\,dv = \frac{1}{2}\int_1^2 ve^w\Big|_{w=-1}^{1}dv = \frac{1}{2}\int_1^2 v(e^1-e^{-1})dv$$

$$=\frac{1}{2}(e-e^{-1})\frac{v^2}{2}\Big|_{v=1}^{2} = \frac{1}{2}(e-e^{-1})\left(\frac{4}{2}-\frac{1}{2}\right) = \boxed{\frac{3}{4}(e-e^{-1})}$$

2. Let $\mathbf{F}=(ye^x+\sin y)\mathbf{i}+(e^x+x\cos y)\mathbf{j}$.

(a) Calculate div($\mathbf{F}$).

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(ye^x+\sin y)+\frac{\partial}{\partial y}(e^x+x\cos y)$$
$$= ye^x - x\sin y$$

(b) Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a potential function f such that $\mathbf{F}=\nabla f$.

$\frac{\partial}{\partial x}(e^x+x\cos y) = e^x+\cos y$

$\frac{\partial}{\partial y}(ye^x+\sin y) = e^x+\cos y$

same $\Rightarrow$ conservative

Need to find f s.t. $\langle f_x, f_y\rangle = \vec{F} \Rightarrow \langle f_x,f_y\rangle = \langle ye^x+\sin y, e^x+x\cos y\rangle$

$f_x = ye^x+\sin y$, $f_y = e^x+x\cos y$ $\Rightarrow$ $f=\int(ye^x+\sin y)dx = ye^x+x\sin y+g(y)$, $f=\int(e^x+x\cos y)dy = ye^x+x\sin y+h(x)$ $\Rightarrow$ $\boxed{f=ye^x+x\sin y+C}$

Lines: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$, $0 \le t \le 1$

$\vec{r}_0$ = starting point

$\vec{v}$ = direction, end point − starting point

Circles: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$

is $x^2+y^2=r^2$ oriented counterclockwise

* Answers will vary

3. Write the parametrization for the following problems. *Hint*: use polar coordinates for circle and disk parametrizations.

(a) The line from $(1, 1)$ to $(4, 2)$

$\vec{v} = \langle 4,2\rangle - \langle 1,1\rangle = \langle 3,1\rangle$

$$\boxed{\begin{aligned}\vec{r}(t) &= \langle 1,1\rangle + t\langle 3,1\rangle \quad 0 \le t \le 1\\ &= \langle 1+3t, 1+t\rangle\end{aligned}}$$

(b) The line from $(4, 2)$ to $(1, 1)$

$\vec{v} = \langle 1,1\rangle - \langle 4,2\rangle = \langle -3,-1\rangle$

$$\boxed{\begin{aligned}\vec{r}(t) &= \langle 4,2\rangle + t\langle -3,-1\rangle \quad 0 \le t \le 1\\ &= \langle 4-3t, 2-t\rangle\end{aligned}}$$

(Another way: $\vec{r}(t) = \langle 1+3t, 1+t\rangle$, $1 \le t \le 0$)

(c) The line from $(2, -4, 5)$ to $(7, 9, -1)$

$\vec{v} = \langle 7,9,-1\rangle - \langle 2,-4,5\rangle$

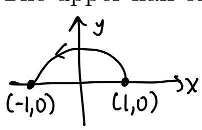
$= \langle 5,13,-6\rangle$

$$\boxed{\begin{aligned}\vec{r}(t) &= \langle 2,-4,5\rangle + t\langle 5,13,-6\rangle \quad 0 \le t \le 1\\ &= \langle 2+5t, -4+13t, 5-6t\rangle\end{aligned}}$$

(d) The arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$

$\vec{r}(x,y) = \langle x,y\rangle$ plug in x : $\boxed{\vec{r}(x) = \langle 4-y^2, y\rangle \quad -3 \le y \le 2}$

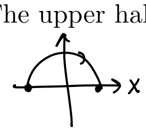
(e) The upper half of the unit circle $x^2 + y^2 = 1$ starting at $(1, 0)$ and ending at $(-1, 0)$



$\vec{r}(x,y) = \langle x,y\rangle$ we know $r=1$

Polar: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle \Rightarrow$ $\boxed{\begin{gathered}\vec{r}(\theta) = \langle \cos\theta, \sin\theta\rangle\\ 0 \le \theta \le \pi\end{gathered}}$

(f) The upper half of the unit circle $x^2 + y^2 = 1$ starting at $(-1, 0)$ and ending at $(1, 0)$



x term moving in opposite direction

$\boxed{\begin{gathered}\vec{r}(\theta) = \langle -\cos\theta, \sin\theta\rangle\\ 0 \le \theta \le \pi\end{gathered}}$

Another way: $\boxed{\begin{gathered}\vec{r}(\theta) = \langle \cos\theta, \sin\theta\rangle\\ \pi \le \theta \le 0\end{gathered}}$

kinda silly looking but useful

(g) The unit circle centered at the origin, oriented clockwise

unit circle means $r=1$ $\boxed{\begin{gathered}\vec{r}(\theta) = \langle -\cos\theta, \sin\theta\rangle\\ 0 \le \theta \le 2\pi\end{gathered}}$

(h) The circle of radius 3 centered at $(-1, 2)$

Circle of radius 3 at origin:

$\vec{r}(\theta) = \langle 3\cos\theta, 3\sin\theta\rangle$

shift to $(-1,2)$:

$\boxed{\begin{gathered}\vec{r}(\theta) = \langle 3\cos\theta - 1, 3\sin\theta + 2\rangle\\ 0 \le \theta \le 2\pi\end{gathered}}$

(i) The circle $x^2 + y^2 = 4$ on the plane $z = -5$, oriented counterclockwise

$\vec{r}(x,y,z) = \langle x,y,z\rangle$

plug in $z=-5$: $\vec{r}(x,y) = \langle x,y,-5\rangle$

polar: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, -5\rangle$

we know $r=2$: $\boxed{\begin{gathered}\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, -5\rangle\\ 0 \le \theta \le 2\pi\end{gathered}}$

(j) The circle $x^2 + z^2 = 9$ on the plane $y = -1$

$\vec{r}(x,y,z) = \langle x,y,z\rangle$

plug in $y=-1$: $\vec{r}(x,z) = \langle x,-1,z\rangle$

polar: $\vec{r}(r,\theta) = \langle r\cos\theta, -1, r\sin\theta\rangle$

$r=3$ $\boxed{\begin{gathered}\vec{r}(\theta) = \langle 3\cos\theta, -1, 3\sin\theta\rangle\\ 0 \le \theta \le 2\pi\end{gathered}}$