SOLUTIONS TO PRACTICE PROBLEMS III

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1. Evaluate the following:

$$\boxed{\ast \int u\,dv = uv - v\,du}$$

(a) $\displaystyle\int \ln x\,dx$ parts $u = \ln x$, $dv = dx$, $du = \frac{1}{x}dx$, $v = x$

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = \boxed{x\ln x - x + C}$$

(b) $\displaystyle\int xe^{-x}dx$ parts $u = x$, $dv = e^{-x}dx$, $du = dx$, $v = -e^{-x}$

$$\int xe^{-x}dx = -xe^{-x} + \int +e^{-x}dx = \boxed{-xe^{-x} - e^{-x} + C}$$

(c) $\displaystyle\int (x^2+1)\cos(4x)dx$ parts $u = x^2+1$, $dv = \cos(4x)\,dx$, $du = 2x$, $v = \frac{1}{4}\sin(4x)$

$$\int (x^2+1)\cos(4x)\,dx = \frac{1}{4}(x^2+1)\sin 4x - \int \cancel{2}x\cdot\frac{1}{\cancel{4}_2}\sin(4x)\,dx$$

parts again $u = \frac{1}{2}x$, $dv = \sin(4x)dx$, $du = dx$, $v = -\frac{1}{4}\cos(4x)$

$$= \frac{1}{4}(x^2+1)\sin 4x - \left(-\frac{1}{8}x\cos(4x) + \int +\frac{1}{4}\cos(4x)\,dx\right)$$

$$= \boxed{\frac{1}{4}(x^2+1)\sin(4x) + \frac{1}{8}x\cos(4x) - \frac{1}{16}\sin(4x) + C}$$

(d) $\displaystyle\int \arctan\left(\frac{1}{x}\right)dx$ parts $u = \arctan\left(\frac{1}{x}\right)$, $dv = dx$, $v = x$

$$du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{-1}{x^2}\,dx = \frac{-1}{x^2+1}\,dx$$

$$\int \arctan\left(\frac{1}{x}\right)dx = x\arctan\left(\frac{1}{x}\right) + \int \frac{+x}{x^2+1}\,dx$$

sub: $w = x^2+1$, $dw = 2x\,dx \Rightarrow \frac{1}{2}dw = x\,dx$, so $\displaystyle\int \frac{x}{x^2+1}dx = \frac{1}{2}\int\frac{1}{w}dw = \frac{1}{2}\ln|w| = \frac{1}{2}\ln(x^2+1)$

$$= \boxed{x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln(x^2+1) + C}$$

(e) $\displaystyle\int x^3e^{-x^2}dx$ sub: $w = -x^2$, $dw = -2x\,dx \Rightarrow \frac{1}{2}dw = -x\,dx$

$$\int x^3e^{-x^2}dx = \int \underbrace{(-x)}\,\underbrace{(-x^2)}_{w}e^{\underbrace{-x^2}_{w}}\,\underbrace{dx}_{\frac{1}{2}dw} = \int \frac{1}{2}we^w\,dw$$

parts $u = \frac{1}{2}w$, $dv = e^w dw$, $du = \frac{1}{2}dw$, $v = e^w$

$$\int \frac{1}{2}we^w\,dw = \frac{1}{2}we^w - \int\frac{1}{2}e^w\,dw = \frac{1}{2}we^w - \frac{1}{2}e^w + C$$

$$= \boxed{-\frac{x^2}{2}e^{-x^2} - \frac{1}{2}e^{-x^2} + C}$$

(f) $\displaystyle\int \frac{x+1}{x^2-2x+2}dx$ complete the square: $x^2-2x+2 = (x^2-2x+\underline{1}) - \underline{1} + 2 = (x-1)^2+1$

$$\int \frac{x+1}{(x-1)^2+1}\,dx$$

$u = x-1 \rightarrow x = u+1$, $du = dx$ $\Rightarrow$

$$\int\frac{x+1}{(x-1)^2+1}\,dx = \int\frac{u+1+1}{u^2+1}\,du = \int\frac{u+2}{u^2+1}\,du$$

$$= \int\frac{u}{u^2+1}\,du + 2\int\frac{1}{u^2+1}\,du = \frac{1}{2}\ln(u^2+1) + 2\tan^{-1}u + C$$

$$\Rightarrow \boxed{\frac{1}{2}\ln\left((x-1)^2+1\right) + 2\tan^{-1}(x-1) + C}$$

(g) $\displaystyle\int \frac{10}{(x-1)(x^2+9)}dx$ partial fractions $\displaystyle\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$

Clear the denominators: $10 = A(x^2+9) + (Bx+C)(x-1)$

$10 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$

$$\underbrace{0x^2}+\underbrace{0x}+\underbrace{10} = (A+B)x^2 + (-B+C)x + 9A - C$$

$$\begin{cases} A+B = 0 \\ -B+C = 0 \\ 9A - C = 10 \end{cases} \quad \text{add} \quad \begin{array}{r} A+B = 0 \\ -B+C=0 \\ \hline A + C = 0 \end{array} \qquad \begin{array}{r} A+C=0 \\ +\,9A - C = 10 \\ \hline 10A = 10 \end{array} \qquad A=1 \quad C=-1 \qquad A+B=0 \Rightarrow B=-1$$

Therefore $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{x-1} + \frac{-1x-1}{x^2+9}\,dx = \int\left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right)dx$

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C \qquad \int \frac{-x}{x^2+9}\,dx = \frac{-1}{2}\ln(x^2+9)+C$$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{9}\int \frac{1}{\frac{x^2}{9}+1}\,dx = \frac{1}{9}\int \frac{3\,du}{u^2+1} = \frac{1}{3}\arctan(u)+C = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C.$$

$u^2 = \frac{x^2}{9}$

$u=\frac{x}{3},\ du=\frac{1}{3}dx$

Therefore $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx = \boxed{\ln|x-1| - \frac{1}{2}\ln(x^2+9) + \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C}$

(h) $\displaystyle\int \frac{1}{x^2-16}dx$ partial fractions $\displaystyle\frac{1}{x^2-16} = \frac{1}{(x-4)(x+4)} = \frac{A}{x-4}+\frac{B}{x+4}$

clear denominators: $1 = A(x+4)+B(x-4)$ at $x=4$: $1=8A \Rightarrow A=1/8$

$x=-4$: $1=-8B \Rightarrow B=-1/8$

$$\int \frac{1}{x^2-16}\,dx = \int\left(\frac{1/8}{x-4} - \frac{1/8}{x+4}\right)dx = \boxed{\frac{1}{8}\ln|x-4| - \frac{1}{8}\ln|x+4|+C}$$

(i) $\displaystyle\int \frac{1}{(x+5)^2(x-1)}dx$ partial fractions $\displaystyle\frac{1}{(x+5)^2(x-1)} = \frac{A}{x+5}+\frac{B}{(x+5)^2}+\frac{C}{x-1}$

Clear the denominators: $1 = A(x+5)(x-1)+B(x-1)+C(x+5)^2$

at $x=1$: $1=36C \Rightarrow C=1/36$

$x=-5$: $1=-6B \Rightarrow B=-1/6$

$x=0$: $1=-5A-B+25C \Rightarrow 1=-5A+\frac{1}{6}+\frac{25}{36} \Rightarrow 1=-5A+\frac{31}{36} \Rightarrow \frac{5}{36}=-5A \Rightarrow A=\frac{-1}{36}$

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int\left(\frac{-1/36}{x+5}+\frac{-1/6}{(x+5)^2}+\frac{1/36}{x-1}\right)dx = \frac{-1}{36}\ln|x+5| - \frac{1}{6}\frac{(x+5)^{-1}}{-1} + \frac{1}{36}\ln|x-1|+C$$
$$= \boxed{\frac{-1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1|+C}$$

2. Determine whether the following improper integrals converge or diverge:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2}dx$ Improper integral because of infinite bounds AND because $\frac{1}{x^2}$ is undefined at $x=0$

$$\int_{-\infty}^{\infty}\frac{1}{x^2}\,dx = 2\int_0^\infty \frac{1}{x^2}\,dx = \frac{-2}{x}\Big|_0^\infty = \lim_{x\to\infty}\left(\frac{-2}{x}\right) - \lim_{x\to0^+}\left(\frac{-2}{x}\right) = 0+\infty=\infty \quad \text{diverges}$$

because $\frac{1}{x^2}$ is even

(b) $\displaystyle\int_5^{10} \frac{1}{x-5}dx$ Integral is improper because $\frac{1}{x-5}$ is undefined at $x=5$

$$\int_5^{10}\frac{1}{x-5}\,dx = \ln|x-5|\Big|_5^{10} = \ln|10-5| - \lim_{x\to5^+}\ln|x-5| = \ln 5 - (-\infty) \quad \text{diverges}$$

(c) $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^3}dx$ Integral is improper because of infinite bounds

$u=\ln x$

$du = \frac{1}{x}\,dx$

at $x=2$, $u=\ln 2$

$x\to\infty$, $u\to\infty$

$$\int_{\ln 2}^{\infty}\frac{1}{u^3}\,du = \frac{u^{-2}}{-2}\Big|_{\ln 2}^{\infty} = \frac{-1}{2u^2}\Big|_{\ln 2}^{\infty} = \lim_{\cancel{x\to\infty}}\left(\frac{-1}{2u^2}\right)^{\to 0} - \left(\frac{-1}{2(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}$$

3. Determine the radius and interval of convergence of the following power series:

(a) $\displaystyle\sum_{n=100}^{\infty} \frac{x^n}{n!}$

Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$ which is always <1

$\boxed{\text{Radius of convergence is } \infty\text{, interval of convergence } (-\infty,\infty)}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{4^n \ln n}$

Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{4^{n+1}\ln(n+1)} \cdot \frac{4^n \ln n}{(-1)^n x^n}\right| = \lim_{n\to\infty} \frac{|x|}{4}\frac{\ln n}{\ln(n+1)} = \frac{|x|}{4}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}$

$\overset{LH}{=} \frac{|x|}{4}\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{|x|}{4}\lim_{n\to\infty}\frac{n+1}{n} = \frac{|x|}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{|x|}{4} < 1$

Therefore $|x|<4 \Rightarrow$ $\boxed{\text{Radius of convergence is } 4}$ $|x|<4 \Rightarrow -4<x<4$ need to test endpoints.

At $x=4$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n 4^n}{4^n \ln n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ Alternating series test: $b_n = \frac{1}{\ln n}$, $\lim_{n\to\infty}\frac{1}{\ln n} = 0$ ✓

$\frac{1}{\ln n}$ is decreasing since for $f(x)=\frac{1}{\ln x}=(\ln x)^{-1}$, $f'(x) = -1(\ln x)^{-2}\cdot\frac{1}{x} < 0$ ✓

Converges by AST, i.e. converges at $x=4$

At $x=-4$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n(-4)^n}{4^n \ln n} = \sum_{n=2}^{\infty}\frac{1}{\ln n}$ Direct Comparison: $\ln n < n \Rightarrow \frac{1}{\ln n} > \frac{1}{n}$. $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges (Harmonic series / p-test w/ $p=1$)

Therefore $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by direct comparison, i.e. diverges at $x=-4$

Interval of convergence is $(-4,4]$

(c) $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)|2x-1| = \infty$ if $x \neq \frac{1}{2}$

never <1, so $\boxed{\text{radius of convergence is } 0 \text{; "interval" of convergence is } \{x=\tfrac{1}{2}\}}$

4. Use the power series $\frac{1}{1-x} = \sum_{n=0}^{n} x^n$, $|x|<1$, to find the power series of the following functions:

(a) $f(x) = \dfrac{1}{x^7-1}$

Goal: make the function look like $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$

$$\frac{1}{x^7-1} = -\left(\frac{1}{\underbrace{1-x^7}_{u=x^7}}\right) = -\frac{1}{1-u} = -\sum_{n=0}^{\infty}u^n = \boxed{-\sum_{n=0}^{\infty}(x^7)^n}$$

(b) $f(x) = \dfrac{x^3}{x+2}$ $= x^3\cdot\frac{1}{2+x} = \frac{x^3}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{x^3}{2}\left(\frac{1}{1-\underbrace{\left(-\frac{x}{2}\right)}_{u}}\right) = \frac{x^3}{2}\left(\frac{1}{1-u}\right) = \frac{x^3}{2}\sum_{n=0}^{\infty}u^n = \frac{x^3}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n$

$$= \frac{x^3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{2^{n+1}}$$

5. Consider the function $f(x) = \frac{1}{x}$, $a=-3$.

(a) Write the expression for the error $|f(-3.1) - T_n(-3.1)|$ where T_n denote the nth Taylor polynomial.

$|f(x) - T_n(x)| \leq \frac{K|x-a|^{n+1}}{(n+1)!}$ where $K = \max_{[a,x]}|f^{(n+1)}(x)|$. Given $a=-3$, $x=-3.1$

Need to find $f^{(n+1)}$.

$f(x) = x^{-1}$

$f'(x) = -1x^{-2}$

$f''(x) = (-1)(-2)x^{-3}$

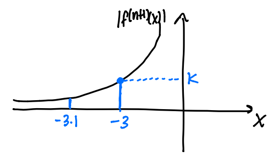
$f'''(x) = (-1)(-2)(-3)x^{-4}$

$\vdots$

$f^{(n)} = (-1)^n n!\, x^{-(n+1)}$

$f^{(n+1)} = (-1)^{n+1}(n+1)!\, x^{-(n+2)}$

Then $|f^{(n+1)}(x)| = \dfrac{(n+1)!}{|x|^{n+2}}$. Now we need to find K.



Plotting the graph helps to see that $|f^{(n+1)}|$ is bigger at $a=-3$,

Therefore $K = \dfrac{(n+1)!}{|-3|^{n+2}} = \dfrac{(n+1)!}{3^{n+2}}$

Therefore $|f(-3.1) - T_n(-3.1)| \le \dfrac{(n+1)!}{3^{n+2}} \dfrac{|-3.1-(-3)|^{n+1}}{(n+1)!} = \dfrac{(0.1)^{n+1}}{3^{n+2}}$

(b) Find the Taylor series for $f(x)$ centered about a.

Taylor series: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ given $a=-3$

In part (a), we found $f^{(n)}(x) = \dfrac{(-1)^n n!}{x^{n+1}} \Rightarrow f^{(n)}(-3) = \dfrac{(-1)^n n!}{(-3)^{n+1}} = \dfrac{(-1)^n n!}{(-1)^{n+1}3^{n+1}} = \dfrac{-(n!)}{3^{n+1}}$

Therefore $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{-(n!)}{3^{n+1}} \cdot \frac{1}{n!}(x-(-3))^n = \boxed{\sum_{n=0}^{\infty} \frac{-1}{3^{n+1}}(x+3)^n}$

6. Use an appropriate Taylor series to evaluate the following:

(a) $\displaystyle\int e^{x^2}dx$ $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

$\displaystyle\int e^{x^2}dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}dx = \int \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots\right)dx = x + \frac{x^3}{3} + \frac{x^5}{2!5} + \frac{x^7}{3!7} + \dots + C$

$= \boxed{\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} + C}$

(b) $\displaystyle\int x\cos(x^3)dx$ $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$

Then $x\cos(x^3) = x\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}$

$\displaystyle\int x\cos(x^3)dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!\, x^{2n}} + C}$

(c) $\displaystyle\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3}$

* you can look up the series for $\tan^{-1}x$ and memorize it, or I will show you how to derive it from scratch (I would at least have the series for e^x, $\sin x$, $\cos x$ memorized)

Deriving the series for $\tan^{-1}x$: Recall that $\tan^{-1}x = \displaystyle\int \frac{1}{1+x^2}dx$. Need to find the series for $\dfrac{1}{1+x^2}$.

$\dfrac{1}{1+x^2} = \dfrac{1}{1-\underbrace{(-x^2)}_{u}} = \dfrac{1}{1-u} = \displaystyle\sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

Then $\tan^{-1}x = \displaystyle\int \sum_{n=0}^{\infty} (-1)^n x^{2n}dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$

Plug in $x=0$ on both sides: $\tan^{-1}0 = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1} + C \Rightarrow 0 = 0 + C \Rightarrow C = 0$

Therefore $\tan^{-1}x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

$\displaystyle\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3} = \lim_{x\to 0} \frac{x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}{x^3} = \lim_{x\to 0} \frac{\cancel{x} - \left(\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)}{x^3}$

$= \displaystyle\lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots}{x^3} = \lim_{x\to 0} \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \dots = \frac{1}{3}$