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Math 31B Worksheet
Week 9

1. Compute each of the following integrals:

$* \int \frac{du}{u} = \ln|u| + C$ (a) $\displaystyle\int \frac{4}{x+8}\,dx = \boxed{4\ln|x+8|+C}$

$*$ Note: can use the substitution $u=x+8$ to evaluate

(b) $\displaystyle\int \frac{3}{(x-7)^3}\,dx = 3\int (x-7)^{-3}dx = \frac{3(x-7)^{-2}}{-2} + C = \boxed{\frac{-3}{2(x-7)^2} + C}$

$*$ Note: can also use the substitution $u = x-7$ to evaluate

$* \int \frac{1}{u^2+1}\,du = \arctan u + C$

$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

(c) $\displaystyle\int \frac{-\frac{1}{3}}{x^2+25}\,dx = -\frac{1}{3}\int \frac{1}{x^2+25}\,dx = -\frac{1}{3}\cdot\frac{1}{25}\int \frac{1}{\frac{x^2}{25}+1}\,dx$ $\quad u^2 = \frac{x^2}{25} \Rightarrow u = \frac{x}{5}$, $du = \frac{1}{5}dx$, $dx = 5du$

$\displaystyle = -\frac{1}{3}\cdot\frac{1}{25}\int \frac{5\,du}{u^2+1} = -\frac{1}{3}\cdot\frac{1}{5}\int \frac{1}{u^2+1}\,du = -\frac{1}{15}\arctan u + C = \boxed{-\frac{1}{15}\arctan\left(\frac{x}{5}\right) + C}$

(d) $\displaystyle\int \frac{-5x}{x^2+4}\,dx = \int \frac{-5\cdot\frac{du}{2}}{u} = -\frac{5}{2}\int \frac{du}{u} = -\frac{5}{2}\ln|u| + C = \boxed{-\frac{5}{2}\ln(x^2+4) + C}$

$u = x^2+4$

$du = 2x\,dx \Rightarrow \frac{du}{2} = x\,dx$

2. For each of the following, write down the partial fraction decomposition, with unknown constants (A, B, C, etc) in the numerators. You do not have to solve for the constants. (I.e., just write down the terms that should be needed for the decomposition.)

(a) $\displaystyle\frac{1}{x(x-1)(x+1)(x+5)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x+5}$

(b) $\displaystyle\frac{1}{(x+1)(x^2+1)(x+9)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{D}{x+9} + \frac{Ex+F}{x^2+9}$

(c) $\displaystyle\frac{1}{x^3(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2}$

(d) $\displaystyle\frac{1}{x(x+4)(x-2)^3(x^2-2x+10)} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3} + \frac{Fx+G}{x^2-2x+10}$

3. Do polynomial long division on $\dfrac{2x^5 - x^4 + 10x^3 - 4x^2}{x^4 + 5x^2 - 36}$ to rewrite it in the form $(quotient) + \frac{(remainder)}{(denominator)}$.

$$
\begin{array}{r}
2x - 1 \\
x^4+0x^3+5x^2+0x-36 \,\big)\, \overline{\cancel{2x^5}-\cancel{x^4}+10x^3-4x^2+0x+0} \\
-\,(\cancel{2x^5}+0x^4+10x^3+0x^2-72x+0) \\
\hline
-\cancel{x^4}+\cancel{0x^3}-4x^2+72x+0 \\
-\,(-\cancel{x^4}+\cancel{0x^3}-5x^2+0x+36) \\
\hline
x^2+72x-36
\end{array}
$$

Therefore $\dfrac{2x^5-x^4+10x^3-4x^2}{x^4+5x^2-36} = 2x-1 + \dfrac{x^2+72x-36}{x^4+5x^2-36}$

4. Compute the integral $\displaystyle\int \frac{1}{x^2(x^2+16)}\,dx$, in the following steps:

(a) Write the partial fraction decomposition of the integrand, as in problem 2.

$$\frac{1}{x^2(x^2+16)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+16}$$

(b) Solve for the constants A, B, C, and D.

Clear the fractions: $x^2(x^2+16)\cdot\dfrac{1}{x^2(x^2+16)} = \left(\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+16}\right)\cdot x^2(x^2+16)$

$$1 = Ax(x^2+16) + B(x^2+16) + (Cx+D)x^2$$

$$1 = Ax^3 + 16Ax + Bx^2 + 16B + Cx^3 + Dx^2$$

$$0x^3+0x^2+0x+1 = (A+C)x^3 + (B+D)x^2 + 16Ax + 16B$$

$$\begin{cases} 0 = A+C \\ 0 = B+D \\ 0 = 16A \longrightarrow A = 0 \\ 1 = 16B \longrightarrow B = +1/16 \end{cases} \qquad \begin{array}{l} C = 0 \\ D = -1/16 \end{array}$$

$$\Rightarrow \frac{1}{x^2(x^2+16)} = \frac{0}{x} + \frac{1/16}{x^2} + \frac{0x - 1/16}{x^2+16}$$

(c) Now rewrite the integral using the partial fractions you found, and compute it.

$$\int \frac{1}{x^2(x^2+16)}\,dx = \int\left(\frac{1/16}{x^2} - \frac{1/16}{x^2+16}\right)dx = \frac{1}{16}\int \frac{1}{x^2}\,dx - \frac{1}{16}\int \frac{1}{x^2+16}\,dx$$

$$= \frac{1}{16}\cdot\frac{-1}{x} - \frac{1}{16}\cdot\frac{1}{16}\int \frac{1}{\frac{x^2}{16}+1}\,dx \qquad u^2 = \frac{x^2}{16} \rightarrow u = \frac{x}{4},\ du = \frac{dx}{4}$$

$$= \frac{-1}{16x} - \frac{1}{16}\cdot\frac{1}{16}\int \frac{4\,du}{u^2+1} = \frac{-1}{16} - \frac{1}{16}\cdot\frac{1}{4}\arctan u + C$$

$$= \boxed{\frac{-1}{16x} - \frac{1}{64}\arctan\left(\frac{x}{4}\right) + C}$$

We have covered how to do a partial fraction decomposition when there is a term like (x^2+16) in the denominator, or even a term like $(x^2-6x+13)$. And you should also know how to integrate something of the form $\dfrac{Ax+B}{x^2+16}$.

But how do you integrate something of the form $\dfrac{Ax+B}{x^2-6x+13}$?

The following problems will answer this question.

5. To motivate this method, let's first review how to integrate something of the form

$$\int \frac{Ax+B}{x^2+16}\,dx$$

Compute this integral, leaving A and B as arbitrary constants. *(Hint: First split it into two separate integrals. Use* ln *for one and* arctan *for the other.)*

$$\int \frac{Ax+B}{x^2+16}\,dx = A\int \frac{x}{x^2+16}\,dx + B\int \frac{1}{x^2+16}\,dx$$

$$\int \frac{x}{x^2+16}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+16) + C$$

$u = x^2+16$
$du = 2x\,dx$

$$\int \frac{1}{x^2+16}\,dx = \frac{1}{16}\int \frac{1}{\frac{x^2}{16}+1}\,dx = \frac{1}{16}\cdot\int \frac{4\,du}{u^2+1} = \frac{1}{4}\arctan u + C = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$$

$u^2 = \frac{x^2}{16} \Rightarrow u = \frac{x}{4}$
$du = \frac{dx}{4}$

$$\Rightarrow \int \frac{Ax+B}{x^2+16}\,dx = \boxed{\frac{A}{2}\ln(x^2+16) + \frac{B}{4}\arctan\left(\frac{x}{4}\right) + C}$$

For the next several problems, suppose we're trying to compute the integral

$$\int \frac{5x-3}{x^2-6x+13}\,dx$$

The trick is to make this look more like the integral you just computed in problem 5. To do that, we need to come up with a clever u-substitution that will make the integrand of this problem look like $\dfrac{Au+B}{u^2+C}$.

We'll focus on the denominator first.

6. Come up with a u so that $x^2 - 6x + 13 = u^2 + (constant)$.
(Hint: First, write

$$x^2 - 6x + 13 = (x^2 - 6x + \underline{\ 9\ }) + \underline{\ 4\ }$$
$$= (x - \underline{\ 3\ })^2 + \underline{\ 4\ }$$

and fill in the blanks appropriately. The last expression gives you a u.)

$x^2 - 6x + 13 = (x^2 - 6x + \underline{9}) - \underline{9} + 13$

take half of this term and square it

$= (x^2 - 6x + 9) + 4 = (x-3)^2 + 4$

Note: You've done this before, although it might have been a long time ago. Do you remember what this process is called? "completing the square"

7. Using the u you came up with in the previous problem, perform a u-substitution on the integral

$$\int \frac{5x-3}{x^2-6x+13}\,dx$$

Note: To turn the x in the numerator into a u, you'll have to use the trick where you solve for x in terms of u and substitute this in place of x. Fortunately, it's quite easy to do that in this case.

$$\int \frac{5x-3}{x^2-6x+13}\,dx = \int \frac{5x-3}{(x-3)^2+4}\,dx = \frac{1}{4}\int \frac{5x-3}{\frac{(x-3)^2}{4}+1}\,dx$$
$$= \frac{1}{4}\int \frac{5(2u+3)-3}{u^2+1}\,2dx = \frac{1}{2}\int \frac{10u+12}{u^2+1}\,du$$

$u^2 = \frac{(x-3)^2}{4} \Rightarrow u = \frac{x-3}{2}$

$du = \frac{1}{2}dx \Rightarrow 2du = dx$

If $u = \frac{x-3}{2} \Rightarrow 2u = x-3 \Rightarrow x = 2u+3$

8. Finally, integrate what you got in the previous problem. Note that it's now exactly the same procedure as in problem 5.

$$\frac{1}{2}\int \frac{10u}{u^2+1} + \frac{12}{u^2+1}\,du = \frac{1}{2}\left(\frac{1}{2}\cdot 10\ln(u^2+1) + 12\arctan u\right) \overset{+C}{=} \frac{5}{2}\ln(u^2+1) + 6\arctan u + C$$
$$= \boxed{\frac{5}{2}\ln\left(\frac{(x-3)^2}{4}+1\right) + 6\arctan\left(\frac{x-3}{4}\right) + C}$$

* Note: $\ln\left(\frac{(x-3)^2}{4}+1\right) = \ln\left(\frac{(x-3)^2+4}{4}\right) = \ln(x^2-6x+13) - \ln(4)$, so another way to write the solution is

$\frac{5}{2}\ln(x^2-6x+13) + 6\arctan\left(\frac{x-3}{4}\right) + C$