Solutions to Practice Problems III

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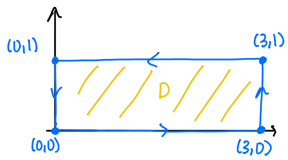
1. Evaluate $\int_C xy\,dx + x^2 dy$ where C is the rectangle with vertices $(0,0), (3,0), (3,1), (0,1)$ oriented counterclockwise.



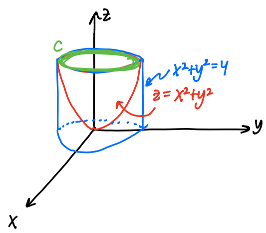
C is closed, we can use Green's Theorem: $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$P = xy \qquad \frac{\partial P}{\partial y} = x \qquad$ Bounds for D: $0 \le x \le 3$

$Q = x^2 \qquad \frac{\partial Q}{\partial x} = 2x \qquad 0 \le y \le 1$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^3 (2x - x)\,dx\,dy = \int_0^1 \int_0^3 x\,dx\,dy = \int_0^3 x\,dx = \frac{x^2}{2}\Big|_0^3 = \frac{9}{2}$$

2. Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$ where $\mathbf{F} = x^2z^2\mathbf{i} + y^2z^2\mathbf{j} + xyz\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies in the cylinder $x^2 + y^2 = 4$.



Stokes' Theorem: $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

Assume positive orientation. Need to find parametrization of C. C is where $z = x^2+y^2$ and $x^2+y^2=4$ intersect, which is the circle $x^2+y^2=4$, $z=4$.

$\vec{r}(x,y,z) = \langle x, y, z\rangle = \langle x, y, 4\rangle \Rightarrow \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4\rangle$

$\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 4\rangle \qquad 0 \le \theta \le 2\pi$

$\vec{r}\,'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0\rangle$

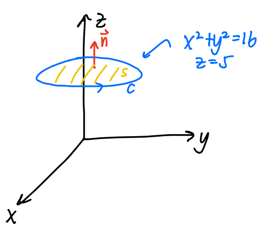
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle (2\cos\theta)^2 4^2, (2\sin\theta)^2 4^2, (2\cos\theta)(2\sin\theta)2\rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0\rangle\,d\theta$$
$$= \int_0^{2\pi} -2\sin\theta \cdot 64\cos^2\theta + 2\cos\theta \cdot 64\sin^2\theta\,d\theta = 128\int_0^{2\pi} (\sin^2\theta\cos\theta - \cos^2\theta\sin\theta)\,d\theta$$

$\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta = \int_0^0 u^2\,du = 0$, also $\int_0^{2\pi} \cos^2\theta\sin\theta\,d\theta = \int_0^0 u^2\,du = 0 \Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$

$u = \sin\theta,\ du = \cos\theta\,d\theta \qquad u = \cos\theta,\ du = -\sin\theta\,d\theta$

3. Consider the vector field $\mathbf{F}(x,y,z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$ where C is the circle $x^2 + y^2 = 16, z = 5$ oriented counterclockwise when viewed from above.

   (a) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by finding an appropriate parametrization vector $\mathbf{r}(t)$.



$\vec{r}(x,y,z) = \langle x, y, z\rangle = \langle x, y, 5\rangle \Rightarrow \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 5\rangle$

$\Rightarrow \vec{r}(\theta) = \langle 4\cos\theta, 4\sin\theta, 5\rangle \qquad 0 \le \theta \le 2\pi$

$\vec{r}\,'(\theta) = \langle -4\sin\theta, 4\cos\theta, 0\rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\,d\theta = \int_0^{2\pi} \langle (4\sin\theta)(5), 2(4\cos\theta)(5), e^{(4\cos\theta)(4\sin\theta)}\rangle \cdot \langle -4\sin\theta, 4\cos\theta, 0\rangle\,d\theta$$
$$= \int_0^{2\pi} (-80\sin^2\theta + 160\cos^2\theta)\,d\theta = \int_0^{2\pi} -\frac{80}{2}(1 - \cos 2\theta) + \frac{160}{2}(1 + \cos 2\theta)\,d\theta$$
$$= -40\left(\theta - \frac{\sin 2\theta}{2}\right) + 80\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = -40(2\pi) + 80(2\pi) = 80\pi$$

   (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem, and verify it is equal to your solution in part (a).

Stokes' says $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S} = \iint_S \text{curl}\,\vec{F} \cdot \vec{n}\,dS$

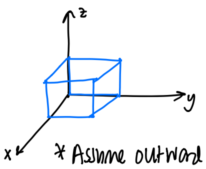
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2xz & e^{xy} \end{vmatrix} = \hat{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2xz & e^{xy} \end{vmatrix} - \hat{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ yz & e^{xy} \end{vmatrix} + \hat{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ yz & 2xz \end{vmatrix} = \langle xe^{xy} - 2x, -(ye^{xy} - y), 2z - z\rangle$$
$$= \langle xe^{xy} - 2x, y - ye^{xy}, z\rangle$$

The normal to the surface is $\vec{n}=\hat{k}$. Plug in what we have so far:

$$\iint_S \langle xe^{xy}-2x,\ y-ye^{xy},\ z\rangle\cdot\langle 0,0,1\rangle\,dS = \iint_S z\,dS$$

But on the disk, $z=5 \Rightarrow \iint_S 5\,dS = 5\iint_S dS = 5\pi(4)^2 = 80\pi$ ✓ same answer we got in (a)!

4. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x,y,z) = 3x\mathbf{i} + xy\mathbf{j} + 2xz\mathbf{k}$ where E is the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$. (*Note*: to verify the theorem is true you need to show calculate both $\iint_S \mathbf{F}\cdot d\mathbf{S}$ and $\iiint_E \operatorname{div}(\mathbf{F})dV$ and show they are equal.)



We will calculate $\iint_S \vec{F}\cdot d\vec{S}$ first. There are 6 faces of the cube:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{top} + \iint_{bottom} + \iint_{left} + \iint_{right} + \iint_{front} + \iint_{back} \vec{F}\cdot d\vec{S}$$

* Assume outward parametrization.

Top of cube: $z=1$, $\vec{n}=\hat{k}$, $\iint_{top} \langle 3x, xy, 2x\cdot 1\rangle\cdot\langle 0,0,1\rangle\,dS = \int_0^1\int_0^1 2x\,dx\,dy = \int_0^1 2x\,dx = x^2\Big|_0^1 = 1$

Bottom of cube: $z=0$, $\vec{n}=-\hat{k}$, $\iint_{bottom} \langle 3x, xy, 2x\cdot 0\rangle\cdot\langle 0,0,-1\rangle\,dS = \int_0^1\int_0^1 0\,dx\,dy = 0$

Left side of cube: $y=0$, $\vec{n}=-\hat{j}$, $\iint_{left} \langle 3x, x\cdot 0, 2xz\rangle\cdot\langle 0,-1,0\rangle\,dS = \int_0^1\int_0^1 0\,dx\,dz = 0$

Right side of cube: $y=1$, $\vec{n}=\hat{j}$, $\iint_{right} \langle 3x, x\cdot 1, 2xz\rangle\cdot\langle 0,1,0\rangle\,dS = \int_0^1\int_0^1 x\,dx\,dz = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

Front of cube: $x=1$, $\vec{n}=\hat{i}$, $\iint_{front} \langle 3, 1\cdot y, 2\cdot 1\cdot z\rangle\cdot\langle 1,0,0\rangle\,dS = \int_0^1\int_0^1 3\,dy\,dz = 3$

Back of cube: $x=0$, $\vec{n}=-\hat{i}$, $\iint_{back} \langle 0,0,0\rangle\cdot\langle -1,0,0\rangle\,dS = \int_0^1\int_0^1 0\,dy\,dz = 0$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = 1+0+0+\tfrac{1}{2}+3+0 = \frac{9}{2}$$

Using Divergence Theorem: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$

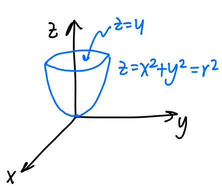
$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(2xz) = 3+x+2x = 3x+3$ bounds for E: $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$

$$\int_0^1\int_0^1\int_0^1 (3x+3)\,dx\,dy\,dz = \int_0^1 (3x+3)\,dx = 3\int_0^1 (x+1)\,dx = \tfrac{3}{2}(x+1)^2\Big|_0^1 = \tfrac{3}{2}\cdot 2^2 - \tfrac{3}{2}\cdot 1^2 = \tfrac{12}{2}-\tfrac{3}{2} = \frac{9}{2}$$ ✓ same as above

5. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = (\cos z + xy^2)\mathbf{i} + xe^{-z}\mathbf{j} + (\sin y + x^2 z)\mathbf{k}$, and S is the surface of the solid bounded by the paraboloid $z = x^2+y^2$ and the plane $z=4$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E (x^2+y^2)\,dV$$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(\cos z + xy^2) + \frac{\partial}{\partial y}(xe^{-z}) + \frac{\partial}{\partial z}(\sin y + x^2 z) = y^2+x^2$



Find bounds using cylindrical coordinates: $0\le r\le 2$, $0\le\theta\le 2\pi$, $r^2\le z\le 4$ $\quad x^2+y^2=r^2$

$$\iiint_E (x^2+z^2)\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta = 2\pi\int_0^2 r^3 z\Big|_{r^2}^4 dr = 2\pi\int_0^2 r^3(4-r^2)\,dr$$

$$= 2\pi\int_0^2 (4r^3-r^5)\,dr = 2\pi\left(r^4-\frac{r^6}{6}\right)\Big|_0^2 = 2\pi\left(16-\frac{64}{6}\right) = 2\pi\left(16-\frac{32}{3}\right)$$

$$= 2\pi\left(\frac{16}{3}\right) = \frac{32\pi}{3}$$