Solutions to Practice Problems I

Written by Victoria Kala, vtkala@math.ucla.edu

1. Circle true (T) or false (F) for each statement below. *Optional challenge: Can you explain/prove why each statement is true or false?*

   (a) T **F** It is possible for a system of linear equations to have exactly three solutions.

   (b) T **F** The following system is consistent:

   $$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

   $$\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right) \xrightarrow{\frac{1}{4}R_3 \to R_3} \left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 1 & -2 & 3 & 1/4 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1/4 \\ 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2}$$

   $$\begin{array}{cccc} -2 & +4 & -6 & -1/2 \\ 2 & -3 & 2 & 1 \\ \hline 0 & 1 & -4 & 1/2 \end{array}$$

   $$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1/4 \\ 0 & 1 & -4 & 1/2 \\ 0 & 1 & -4 & 8 \end{array}\right) \xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1/4 \\ 1 & -4 & 1/2 & 0 \\ 0 & 0 & 0 & 15/2 \end{array}\right) \leftarrow \text{no solution}$$

   (c) **T** F The following system has a unique solution:

   $$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_2 &= 8 \longrightarrow -6x_2 = 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

   $$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -6 & 0 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right) \xrightarrow[\frac{1}{5}R_3 \to R_3]{-\frac{1}{6}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4/3 \\ 1 & 0 & -1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -4/3 \\ 1 & -2 & 1 & 0 \end{array}\right) \xrightarrow{-R_1 + R_3 \to R_3}$$

   $$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -4/3 \\ 0 & -2 & 2 & -2 \end{array}\right) \xrightarrow{2R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 2 & -14/3 \end{array}\right)$$

   $$\begin{array}{cccc} 0 & 2 & 0 & -8/3 \\ 0 & -2 & 2 & -2 \\ \hline 0 & 0 & 2 & -14/3 \end{array}$$

   not in rref BUT it will have rank 3 $\Rightarrow$ unique sol

   (d) T **F** The vector $(3, -1)$ is a solution to the system $A\mathbf{x} = \mathbf{b}$ where

   $$A = \begin{pmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$$

   $$\begin{pmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 2(-1) \\ -2(3) + 5(-1) \\ -5(3) + 6(-1) \end{pmatrix} = \begin{pmatrix} 3-2 \\ -6-5 \\ -15-6 \end{pmatrix} \neq \vec{b}$$

   (e) **T** F Any system with a 3×3 matrix with rank 3 has a unique solution.

   (f) T **F** Any system with a 3×5 matrix with rank 3 has a unique solution.

   (g) T **F** Any system with a 4×3 matrix with rank 3 has a unique solution.

   (h) T **F** For any two matrices A, B, the sum $A + B$ is always defined.

   (i) T **F** All transformations are linear transformations.

   (j) T **F** Every transformation $T(x)$ can be written as $T(x) = Ax$ for some matrix A.

   (k) **T** F Every linear transformation $T(x)$ can be written as $T(x) = Ax$ for some matrix A.

   (l) **T** F The transformation $T(\mathbf{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$ projects $\mathbf{x} = (x, y)$ onto the y-axis.

   (m) T **F** If A and B are square matrices, then $AB = BA$.

   (n) T **F** For any two matrices A, B, the product AB is always defined.

   (o) T **F** All square matrices have an inverse.

   (p) T **F** If A and B are invertible matrices then $(AB)^{-1} = A^{-1}B^{-1}$.

   (q) **T** F If A and B are invertible matrices then $(AB)^{-1} = B^{-1}A^{-1}$.

2. Consider the system

$$\begin{aligned} x_1 + \quad\;\; + x_3 &= 2 \\ + x_2 + \alpha x_3 &= 0 \\ x_1 + 2x_2 + 13x_3 &= 0 \end{aligned}$$

(a) For what values of α is the system

i. Inconsistent?

ii. Consistent?

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & \alpha & 0 \\ 1 & 2 & 13 & 0 \end{array}\right) \quad R_3 - R_1 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & \alpha & 0 \\ 0 & 2 & 12 & -2 \end{array}\right) \quad -2R_2 + R_3 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 12-2\alpha & -2 \end{array}\right)$$

$$\begin{array}{cccc} 0 & -2 & -2\alpha & 0 \\ 0 & 2 & 12 & -2 \\ \hline 0 & 0 & 12-2\alpha & -2 \end{array}$$

If $12-2\alpha = 0$ then no solution

If $12-2\alpha \neq 0$ then at least one solution

$\boxed{\text{(i) } \alpha = 6 \quad \text{(ii) } \alpha \neq 6}$

(b) Is it possible to find a value of α such that the system has infinitely many solutions? Why or why not?

No. If $\alpha \neq 6$ then we have a 3×3 system w/ rank 3 $\Rightarrow$ unique solution

(c) For $\alpha = 7$, find the solution of the system using Gauss-Jordan elimination.

In (a) we found $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 12-2\alpha & -2 \end{array}\right)$ Plug in $\alpha = 7$: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right)$ $-\frac{1}{2}R_3 \to R_3$ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$ $R_1 - R_3 \to R_3$, $R_2 - 7R_3 \to R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad \boxed{\vec{x} = \begin{pmatrix} 1 \\ -7 \\ 1 \end{pmatrix}}$$

Check:

$x_1 + x_3 = 2$ $\quad 1 + 1 = 2$ ✓

$+x_2 + 7x_3 = 0$ $\quad -7 + 7 \cdot 1 = 0$ ✓

$x_1 + 2x_2 + 13x_3 = 0$ $\quad 1 + 2(-7) + 13 \cdot 1 = 0$ ✓

3. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$.

(a) Find $3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = 3\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-4-3 \\ 0+2-1 \\ 9+0+3 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 12 \end{pmatrix}$

(b) Can $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix}$ be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_3$? If so, find it.

$$\begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \to \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 3 & 0 & 3 & 9 \end{array}\right) \quad -3R_1 + R_3 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 6 & -6 & 6 \end{array}\right) \quad -6R_2 + R_3 \to R_3$$

$$\begin{array}{cccc} -3 & 6 & -9 & -3 \end{array}$$

goal: find a, b, c

$$\begin{array}{cccc} 0 & 2 & -2 & 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad 2R_2 + R_1 \to R_1 \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

∞ many solutions

$a + c = 3 \to a = 3 - t$

$b - c = 1 \to b = 1 + t$

let $c = t \in \mathbb{R}$

Yes, $\vec{b_1}$ is linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$ and
$\vec{b_1} = (3-t)\vec{v_1} + (1+t)\vec{v_2} + t\vec{v_3}$
where $t \in \mathbb{R}$.

(c) Can $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$ be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_3$? If so, find it.

$$\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -4 \\ 3 & 0 & 3 & 5 \end{array}\right) \begin{array}{c} \\ \\ -3R_1+R_3 \to R_3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 6 & -6 & 2 \end{array}\right) -6R_2+R_3 \to R_3$$

$-3 \quad 6 \quad -9 \quad -3$

goal: find a, b, c

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 26 \end{array}\right) \to 0 \neq 26 \qquad \text{No solution}$$

$\boxed{\text{No, } \vec{b} \text{ is not a linear combination of } \vec{v}_1, \vec{v}_2, \vec{v}_3}$

4. Let T be the transformation defined by the formula

$$T(x_1, x_2, x_3) = (x_2, -x_1, x_1 + 3x_2, x_1 - x_3)$$

(a) Show T is linear.

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + T\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ x_1+3x_2 \\ x_1-x_3 \end{pmatrix} + \begin{pmatrix} y_2 \\ -y_1 \\ y_1+3y_2 \\ y_1-y_3 \end{pmatrix}$$

$$T\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = \begin{pmatrix} x_2+y_2 \\ -(x_1+y_1) \\ (x_1+y_1)+3(x_2+y_2) \\ x_1+y_1-(x_3+y_3) \end{pmatrix}$$

same

i.e. $T(\vec{x}) + T(\vec{y}) = T(\vec{x}+\vec{y})$

Let $a \in \mathbb{R}$. $aT\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a\begin{pmatrix} x_2 \\ -x_1 \\ x_1+3x_2 \\ x_1-x_3 \end{pmatrix}$

$$T\begin{pmatrix} ax_1 \\ ax_2 \\ ax_3 \end{pmatrix} = \begin{pmatrix} ax_2 \\ -ax_1 \\ ax_1+3ax_2 \\ ax_1-ax_3 \end{pmatrix}$$

same

i.e. $T(a\vec{x}) = aT(\vec{x})$

T is linear

(b) Find a matrix A for the linear transformation such that $T(\mathbf{x}) = A\mathbf{x}$.

$$A = \left(T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \qquad T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1+3\cdot 0 \\ 1-0 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -0 \\ 0+3\cdot 1 \\ 0-0 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -0 \\ 0+3\cdot 0 \\ 0-1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad \text{i.e. } \boxed{T(\vec{x}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix} \vec{x}}$$

$$\text{* Check: } \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0+x_2+0 \\ -x_1+0+0 \\ x_1+3x_2+0 \\ x_1+0-x_3 \end{pmatrix} = T(\vec{x}) \checkmark$$

(c) Find $T(2, -1, 0)$.

One way: plug directly into T: $T\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 2+3(-1) \\ 2-0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \\ 2 \end{pmatrix}$

same

another way: Use (b), i.e. Find $A\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-1+0 \\ -2+0+0 \\ 2-3+0 \\ 2+0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \\ 2 \end{pmatrix}$

5. The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $(1, 2)$ to $(-1, 1)$ and $(0, -1)$ to $(2, -1)$ will map $(1, 1)$ to what value?

Given $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Need to find $T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Since T is linear, $T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$

In particular, $T\left(a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = aT\begin{pmatrix} 1 \\ 2 \end{pmatrix} + bT\begin{pmatrix} 0 \\ -1 \end{pmatrix} \to T\left(\underbrace{a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 0 \\ -1 \end{pmatrix}}\right) = a\begin{pmatrix} -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

These are the two given vectors

Q: Can I find a, b so that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 0 \\ -1 \end{pmatrix}$?

⑤ continued...

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & -1 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -1 \end{array}\right) \xrightarrow{-R_2\to R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) \quad \text{i.e. } a=1,\ b=1.$$

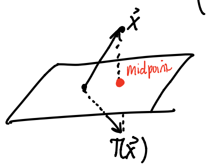
Then $T\left(1\begin{pmatrix}1\\2\end{pmatrix}+1\begin{pmatrix}0\\-1\end{pmatrix}\right) = 1\begin{pmatrix}-1\\1\end{pmatrix}+1\begin{pmatrix}2\\-1\end{pmatrix}$

$$T\begin{pmatrix}1\\1\end{pmatrix}=\boxed{\begin{pmatrix}1\\0\end{pmatrix}}$$

6. Let $T(\mathbf{x})$ be the linear transformation that reflects $\mathbf{x}$ about the plane $-x_1 + 3x_2 + 2x_3 = 0$.

(a) Find $T(1, 4, 0)$. * $T(\vec{x}) = \vec{x} - 2\,\text{proj}_{\vec{n}}(\vec{x})$ where $\vec{n} = (-1, 3, 2)$

$$T\begin{pmatrix}1\\4\\0\end{pmatrix} = \begin{pmatrix}1\\4\\0\end{pmatrix} - 2\frac{\begin{pmatrix}1\\4\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}{\begin{pmatrix}-1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\4\\0\end{pmatrix} - 2\frac{(-1+12+0)}{1+9+4}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\4\\0\end{pmatrix} - \frac{11}{7}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}1+11/7\\4-33/7\\-22/7\end{pmatrix} = \boxed{\begin{pmatrix}18/7\\-5/7\\-22/7\end{pmatrix}}$$



* One way to check the answer is to find the "midpoint" of the two vectors and check it is on the plane.

"Midpoint" $= \begin{pmatrix}1\\4\\0\end{pmatrix} + \begin{pmatrix}18/7\\-5/7\\-22/7\end{pmatrix} = \begin{pmatrix}25/7\\23/7\\-22/7\end{pmatrix}$ $\quad -\left(\frac{25}{7}\right) + 3\left(\frac{23}{7}\right) + 2\left(\frac{-22}{7}\right) = \frac{-25+69-44}{7} = 0$ ✓

(b) Find the matrix representation for $T(\mathbf{x})$.

$$A = \left(T\begin{pmatrix}1\\0\\0\end{pmatrix}\ \ T\begin{pmatrix}0\\1\\0\end{pmatrix}\ \ T\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} - 2\frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}{\begin{pmatrix}-1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} - 2\frac{(-1+0+0)}{1+9+4}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{7}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}6/7\\3/7\\2/7\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - 2\frac{\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}{\begin{pmatrix}-1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - 2\frac{(0+3+0)}{1+9+4}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{3}{7}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}3/7\\-2/7\\-6/7\end{pmatrix}$$

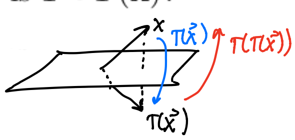
$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} - 2\frac{\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}{\begin{pmatrix}-1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} - 2\frac{(0+0+2)}{1+9+4}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{2}{7}\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}2/7\\-6/7\\3/7\end{pmatrix}$$

$$A = \frac{1}{7}\begin{pmatrix}6 & 3 & 2\\3 & -2 & -6\\2 & -6 & 3\end{pmatrix} \qquad \text{i.e.} \quad \boxed{T(\vec{x}) = \frac{1}{7}\begin{pmatrix}6 & 3 & 2\\3 & -2 & -6\\2 & -6 & 3\end{pmatrix}\vec{x}}$$

* We can use (a) to check: $\frac{1}{7}\begin{pmatrix}6 & 3 & 2\\3 & -2 & -6\\2 & -6 & 3\end{pmatrix}\begin{pmatrix}1\\4\\0\end{pmatrix} = \frac{1}{7}\begin{pmatrix}6+12+0\\3-8+0\\2-24+0\end{pmatrix} = \frac{1}{7}\begin{pmatrix}18\\-5\\-22\end{pmatrix}$ ✓

(c) Without calculating directly, what is $T \circ T(\mathbf{x})$?

reflect about the plane, then reflect again



$T(T(\vec{x})) = \vec{x}$.

(d) Without calculating directly, what is $T \circ T \circ T(\mathbf{x})$?

reflect about the plane 3 times $\rightarrow$ $T(T(T(\vec{x}))) = T(\vec{x})$.

7. Let $B = \begin{pmatrix}4 & -1\\ 0 & 2\end{pmatrix}$. Compute the following:

(a) $B^2 - 2B + I$

$$B^2 = \begin{pmatrix}4&-1\\0&2\end{pmatrix}\begin{pmatrix}4&-1\\0&2\end{pmatrix} = \begin{pmatrix}16+0 & -4-2\\ 0+0 & 0+4\end{pmatrix} = \begin{pmatrix}16&-6\\0&4\end{pmatrix}$$

$$B^2-2B+I = \begin{pmatrix}16&-6\\0&4\end{pmatrix} - 2\begin{pmatrix}4&-1\\0&2\end{pmatrix} + \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}16-8+1 & -6+2+0\\ 0-0+0 & 4-4+1\end{pmatrix} = \boxed{\begin{pmatrix}9&-4\\0&1\end{pmatrix}}$$

(b) $B^{-3} = (B^{-1})^3$

$$B^{-1} = \frac{1}{4\cdot 2 - 0(-1)}\begin{pmatrix}2&+1\\0&4\end{pmatrix} = \frac{1}{8}\begin{pmatrix}2&-1\\0&4\end{pmatrix}$$

$$\begin{pmatrix}4&-6\\0&16\end{pmatrix}$$

$$(B^{-1})^3 = \frac{1}{8}\begin{pmatrix}2&-1\\0&4\end{pmatrix}\frac{1}{8}\begin{pmatrix}2&-1\\0&4\end{pmatrix}\frac{1}{8}\begin{pmatrix}2&-1\\0&4\end{pmatrix} = \frac{1}{8^3}\begin{pmatrix}2&-1\\0&4\end{pmatrix}\begin{pmatrix}2&-1\\0&4\end{pmatrix}\begin{pmatrix}2&-1\\0&4\end{pmatrix} = \frac{1}{8^3}\begin{pmatrix}4+0&-2-4\\0+0&0+16\end{pmatrix}\begin{pmatrix}2&-1\\0&4\end{pmatrix} = \frac{1}{8^3}\begin{pmatrix}8+0&-4-24\\0+0&0+64\end{pmatrix}$$

$$= \boxed{\frac{1}{8^3}\begin{pmatrix}8&-28\\0&64\end{pmatrix}}$$

8. If the matrices

$$\overbrace{\begin{pmatrix}3&-2&-2\\-1&1&1\\3&-1&-2\end{pmatrix}}^{A} \quad\text{and}\quad \overbrace{\begin{pmatrix}1&a&0\\-1&b&1\\2&c&-1\end{pmatrix}}^{B}$$

* a lot of ways to do this problem

are inverse of each other, what is the value of c?

A, B inverses means $AB = BA = I$.

So $$I = BA = \begin{pmatrix}1&a&0\\-1&b&1\\2&c&-1\end{pmatrix}\begin{pmatrix}3&-2&-2\\-1&1&1\\3&-1&-2\end{pmatrix} = \begin{pmatrix}3-a+0 & -2+a+0 & -2+a+0\\ -3-b+3 & 2+b-1 & 2+b-2\\ 6-c-3 & -4+c+1 & -4+c+2\end{pmatrix} = \begin{pmatrix}3-a & -2+a & -2+a\\ -b & b+1 & b\\ 3-c & c-3 & c-2\end{pmatrix}$$

i.e. $$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} = \begin{pmatrix}3-a & -2+a & -2+a\\ -b & b+1 & b\\ 3-c & c-3 & c-2\end{pmatrix}$$

Need $a=2$, $b=0$, $\boxed{c=3}$

9. (a) Calculate the inverse of

$$\begin{pmatrix}3&4&-1\\1&0&3\\2&5&-4\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc}3&4&-1&1&0&0\\1&0&3&0&1&0\\2&5&-4&0&0&1\end{array}\right) \xrightarrow{R_1 \Leftrightarrow R_2} \left(\begin{array}{ccc|ccc}1&0&3&0&1&0\\3&4&-1&1&0&0\\2&5&-4&0&0&1\end{array}\right) \begin{array}{l}-3R_1+R_2 \to R_2\\ -2R_1+R_3\to R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&4&-10&1&-3&0\\0&5&-10&0&-2&1\end{array}\right) R_2 - R_3 \to R_2$$

$$\left(\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&-1&0&1&-1&-1\\0&5&-10&0&-2&1\end{array}\right) 5R_2+R_3\to R_3$$

$$\left(\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&-1&0&1&-1&-1\\0&0&-10&+5&-7&-4\end{array}\right) \begin{array}{l}-R_2\to R_2\\ -\frac{1}{10}R_3\to R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&1&0&-1&1&1\\0&0&1&-5/10&7/10&4/10\end{array}\right) -3R_3+R_1\to R_1$$

$$\left(\begin{array}{ccc|ccc}1&0&0&+15/10&-11/10&-12/10\\0&1&0&-1&1&1\\0&0&1&-5/10&7/10&4/10\end{array}\right)$$

$$\boxed{\text{inverse is } \frac{1}{10}\begin{pmatrix}15&-11&-12\\-10&10&10\\-5&7&4\end{pmatrix}}$$

* Check: $$\begin{pmatrix}3&4&-1\\1&0&3\\2&5&-4\end{pmatrix}\frac{1}{10}\begin{pmatrix}+15&-11&-12\\-10&10&10\\-5&7&4\end{pmatrix} = \frac{1}{10}\begin{pmatrix}45-40+5 & -33+40-35 & -36+40-4\\ 15+0-15 & -11+0+21 & -12+0+12\\ 30-50+20 & -22+50-28 & -24+50-16\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} ✓$$

(b) Use what you found in part (a) to solve the system

$$\begin{aligned}3x_1 + 4x_2 - x_3 &= 1\\ x_1 + \quad +3x_3 &= 0\\ 2x_1 + 5x_2 - 4x_3 &= 2\end{aligned}$$

$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

$$\vec{x} = \frac{1}{10}\begin{pmatrix}15&-11&-12\\-10&10&10\\-5&7&4\end{pmatrix}\begin{pmatrix}1\\0\\2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}15-24\\-10+20\\-5+8\end{pmatrix} = \boxed{\frac{1}{10}\begin{pmatrix}-9\\10\\3\end{pmatrix}}$$