Math 33A − Week 2

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Name: KEY

1. Let $A = \begin{pmatrix} 1 & 5 & 3 \\ 0 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix}$. Compute the following if defined. If not defined, state why.

   (a) $B + C$

   undefined, B and C are not the same size

   (b) $A - 2B$

   $\begin{pmatrix} 1 & 5 & 3 \\ 0 & -1 & 2 \end{pmatrix} - 2\begin{pmatrix} 0 & -3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

   $\begin{pmatrix} 1 & 5 & 3 \\ 0 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & -6 & 2 \\ 2 & 0 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 11 & 1 \\ -2 & -1 & 0 \end{pmatrix}}$

2. Give examples of a matrix A with rank 2 and appropriate sized vector b so that the system $Ax = b$ has * Answers will vary

   (a) No solution

   $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{array}\right)$ $0 \neq 5$ no solution

   A b

   (b) One solution

   $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$

   unique sol is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   (c) Infinitely many solutions

   $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$

   ↑ missing pivot

   solution is $\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$, $t \in \mathbb{R}$

3. Let $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $y = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$. Compute $x \cdot y$.

   $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} = 1 \cdot 4 + 2(-5) + 3 \cdot 6 = 4 - 10 + 18 = 12$

4. Is $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of $y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $z = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$?

i.e. is there a, b s.t. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$?

$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right)$ ← no solution

No

5. Determine which of the following transformations are linear. If the transformation is linear, find the matrix A so that $T(x) = Ax$.

(a) $T(x_1) = x_1^2$

(b) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$

(c) $T(x_1,\ x_2,\ x_3) = (x_1 + 1,\ 5x_2 - x_3)$

(d) $T(x_1,\ x_2) = (x_1,\ 5x_2,\ x_1 - 3x_2,\ 3x_2 - x_1)$

(a) $\left.\begin{array}{l} T(x)+T(y) = x^2+y^2 \\ T(x+y) = (x+y)^2 \end{array}\right\}$ not the same

also $\left.\begin{array}{l} T(ax) = (ax)^2 \\ aT(x) = ax^2 \end{array}\right\}$ not always the same

Not linear

(b) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ x_1-x_2 \end{pmatrix}+\begin{pmatrix} y_1+y_2 \\ y_1-y_2 \end{pmatrix}$

$T\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} = \begin{pmatrix} (x_1+y_1)+(x_2+y_2) \\ (x_1+y_1)-(x_2+y_2) \end{pmatrix}$ same

(b) continued

$\left.\begin{array}{l} aT\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} x_1+x_2 \\ x_1-x_2 \end{pmatrix} \\ T\begin{pmatrix} ax_1 \\ ax_2 \end{pmatrix} = \begin{pmatrix} ax_1+ax_2 \\ ax_1-ax_2 \end{pmatrix} \end{array}\right\}$ same

T is linear, $A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

i.e. $T(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\vec{x}$

(c) $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}+T\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \left(\begin{pmatrix} x_1+1 \\ 5x_2-x_3 \end{pmatrix}+\begin{pmatrix} y_1+1 \\ 5y_2-y_3 \end{pmatrix}\right)$ not the same

$T\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = \begin{pmatrix} (x_1+y_1)+1 \\ 5(x_2+y_2)-(x_3+y_3) \end{pmatrix}$

Also $aT\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a\begin{pmatrix} x_1+1 \\ 5x_2-x_3 \end{pmatrix} \leftarrow$ not the same $\rightarrow T\begin{pmatrix} ax_1 \\ ax_2 \\ ax_3 \end{pmatrix} = \begin{pmatrix} ax_1+1 \\ 5(ax_2)-ax_3 \end{pmatrix}$

Not linear

(d) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 5x_2 \\ x_1-3x_2 \\ 3x_2-x_1 \end{pmatrix}+\begin{pmatrix} y_1 \\ 5y_2 \\ y_1-3y_2 \\ 3y_2-y_1 \end{pmatrix} \leftarrow$ same $\rightarrow T\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ 5(x_2+y_2) \\ (x_1+y_1)-3(x_2+y_2) \\ 3(x_2+y_2)-(x_1+y_1) \end{pmatrix}$

$T\begin{pmatrix} ax_1 \\ ax_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ 5(ax_2) \\ ax_1-3ax_2 \\ 3ax_2-ax_1 \end{pmatrix} \leftarrow$ same $\rightarrow aT\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} x_1 \\ 5x_2 \\ x_1-3x_2 \\ 3x_2-x_1 \end{pmatrix}$ T is linear

$A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 5 \\ 1 & -3 \\ -1 & 3 \end{pmatrix}$

6. (a) Let $x = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$. Find a, b, c so that

$$x = a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 5 \end{array}\right)$ $\begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 1 & 4 \end{array}\right)$ $R_3 - R_2 \to R_3$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right)$ $\boxed{\begin{array}{l} a=1 \\ b=-3 \\ c=7 \end{array}}$

Check: $1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-0+0 \\ 1-3+0 \\ 1-3+7 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ ✓

(b) Suppose we know T is a linear transformation with $T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$. What is $T(x)$?

* T linear means $T(a\vec{x}+b\vec{y}) = aT(\vec{x})+bT(\vec{y})$

From (a), $x = a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$T(x) = T\left(a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

$= aT\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + cT\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= a\begin{pmatrix} -6 \\ 2 \end{pmatrix} + b\begin{pmatrix} -7 \\ 4 \end{pmatrix} + c\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

But from (a) we also know $a=1, b=-3, c=7$

$\Rightarrow T(x) = 1\begin{pmatrix} -6 \\ 2 \end{pmatrix} - 3\begin{pmatrix} -7 \\ 4 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} -6 \\ 2 \end{pmatrix} + \begin{pmatrix} 21 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 28 \end{pmatrix}$

$= \begin{pmatrix} 15 \\ 34 \end{pmatrix}$