Math 33B Worksheet 1 (Integrating Factors)

Name: KEY Score: ____________

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Circle the day of your discussion: Tuesday Thursday

1. Find the general solution to

$$y' = -3\frac{y}{x} + \frac{\sqrt{1+x^2}}{x^2}$$

$y' + \frac{3y}{x} = \frac{\sqrt{1+x^2}}{x^2}$ 1st order linear

Integrating factor: $\mu = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$x^3\left(y' + \frac{3y}{x}\right) = x^3\left(\frac{\sqrt{1+x^2}}{x^2}\right)$

$x^3y' + 3x^2y = x\sqrt{1+x^2}$

$x^3y' + (x^3)'y = x\sqrt{1+x^2}$

$(x^3y)' = x\sqrt{1+x^2}$

$x^3y = \int x\sqrt{1+x^2}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$u = 1+x^2$

$du = 2x\,dx \rightarrow \frac{du}{2} = x\,dx$

$= \frac{1}{3}(1+x^2)^{3/2} + C$

$$\boxed{y = \frac{\frac{1}{3}(1+x^2)^{3/2} + C}{x^3}}$$

2. Find the general solution to

$$\ln(t-1)y' + \frac{y}{t-1} = (t-1)\cos((t-1)^2)$$

Notice that $(\ln(t-1))' = \frac{1}{t-1} \Rightarrow$ $\ln(t-1)y' + (\ln(t-1))'y = (t-1)\cos((t-1)^2)$ i.e. integrating factor has already been multiplied through

$(\ln(t-1)\,y)' = (t-1)\cos((t-1)^2)$

$\ln(t-1)y = \int (t-1)\cos((t-1)^2)\,dt = \int \frac{1}{2}\cos(u)\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin((t-1)^2) + C$

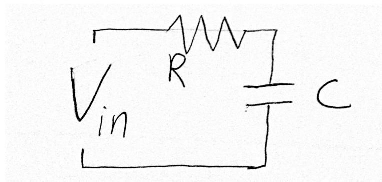
$u = (t-1)^2$

$du = 2(t-1)\,dt \rightarrow \frac{du}{2} = (t-1)dt$

$$\boxed{y = \frac{\frac{1}{2}\sin((t-1)^2) + C}{\ln(t-1)}}$$

3. **Simple Circuitry (TLDR in the next paragraph):** Some of the simplest electronic components are called resistors and capacitors. Resistors, as the name implies, resist the flow of current according to Ohm's law, $V = IR$, where V is the voltage across a resistor, I is the current flowing through it, and R is the resistance of the resistor. Capacitors, on the other hand, are a means of storing both charge and energy. As current flows into them, they accumulate charge, which results in a voltage across them of $V = Q/C$, where Q is the charge, and C is the capacitance (the C is in the denominator since something with a large capacity to hold charge can't quickly change their voltage). It is important to

note that the current in a circuit is derivative of the amount of charge that has flowed through it, i.e. $I = Q'$.



TLDR: We can use these simple laws, along with Kirchhoff's loop rule (the total voltage around any circle must always be zero) to see that the simple RC circuit above must obey the ODE:

$$V_{in}(t) = RQ' + \frac{Q}{C}$$

Where V_{in} is the voltage we're applying to the circuit, for instance by attaching it to an outlet or battery. Let's see how this behaves in a few different circumstances.

(a) What happens if we have no applied voltage (i.e. $V_{in} = 0$) and $Q(0) = 1$? How does this depend on R and C (e.g. what happens when each becomes very large, or very small).

$V=0 \Rightarrow RQ' + \frac{Q}{C} = 0$ * both separable & 1st order linear, your choice on how to solve

Separable: $R\frac{dQ}{dt} = -\frac{Q}{C}$

$\frac{dQ}{Q} = \frac{-1}{RC}\,dt$

$\int \frac{dQ}{Q} = \int \frac{-1}{RC}\,dt$

$\ln|Q| = \frac{-1}{RC}t + a$

$Q = e^{-t/RC}e^{a}$

$Q = Ae^{-t/RC}$

when $t=0, Q=1$: $1 = Ae^{0} \Rightarrow A=1$

$\boxed{Q = e^{-t/RC}}$

If R,C large, $Q \to e^{0} = 1$

If R,C small, $Q \to e^{\text{"}-\infty\text{"}} = 0$.

(b) What happens when we attach a 9-volt battery, so that $V_{in}(t) = 9$? How does this new result depend on R and C? What is the $t \to \infty$ limit?

$V=9 \Rightarrow RQ' + \frac{Q}{C} = 9$ * both separable & 1st order linear, your choice on how to solve

1st order linear: $Q' + \frac{Q}{RC} = \frac{9}{R}$

$\mu = e^{\int \frac{1}{RC}dt} = e^{t/RC}$

$e^{t/RC}\left(Q' + \frac{Q}{RC}\right) = \frac{9}{R}e^{t/RC}$

$(e^{t/RC}Q)' = \frac{9}{R}e^{t/RC}$

$e^{t/RC}Q = \int \frac{9}{R}e^{t/RC}\,dt$

$e^{t/RC}Q = 9Ce^{t/RC} + a$

$\boxed{Q = 9C + ae^{-t/RC}}$

$Q \to 9C$ as $t \to \infty$

(c) Why does this limit make sense?

As $t \to \infty$, $\frac{Q}{C} = V \Rightarrow Q = 9C$ since $V_{in} = 9$

(d) What is the general solution if we plug an RC circuit into an outlet, with AC current (say $V_{in} = \cos(t)$).

$V = \cos t \Rightarrow \quad RQ' + \frac{Q}{C} = \cos t \qquad$ * 1st order linear

$Q' + \frac{Q}{RC} = \frac{1}{R}\cos t$

$\mu = e^{\int \frac{1}{RC} dt} = e^{t/RC}$

$e^{t/RC}\left(Q' + \frac{Q}{RC}\right) = \frac{1}{R} e^{t/RC} \cos t$

$\left(e^{t/RC} Q\right)' = \frac{1}{R} e^{t/RC} \cos t$

$e^{t/RC} Q = \frac{1}{R}\int e^{t/RC} \cos t \, dt \longrightarrow$

parts: $\int e^{t/RC} \cos t \, dt = RC e^{t/RC} \cos t + \int RC e^{t/RC} \sin t \, dt$

$u = \cos t \quad dv = e^{t/RC} dt \qquad u = \sin t \quad dv = RCe^{t/RC} dt$

$du = -\sin t dt \quad v = RCe^{t/RC} \qquad du = \cos t dt \quad v = (RC)^2 e^{t/RC}$

$= RCe^{t/RC}\cos t + (RC)^2 e^{t/RC} \sin t - \int (RC)^2 e^{t/RC} \cos t dt$

$(1+(RC)^2)\int e^{t/RC}\cos t dt = RC e^{t/RC}\cos t + (RC)^2 e^{t/RC}\sin t$

$\int e^{t/RC} \cos t \, dt = \frac{1}{1+(RC)^2} e^{t/RC}\left(RC\cos t + (RC)^2 \sin t\right) + a$

$$\boxed{Q = \frac{1}{1+(RC)^2}\left(C\cos t + RC^2 \sin t\right) + ae^{-t/RC}}$$

(e) In this case, the charge Q and the applied voltage do not reach maximums at the same time. Assuming $R = C = 1$, and ignoring any exponential terms, how long after the maximum voltage occurs does the maximum charge occur? How do you think this would change with R and C (increase, decrease, stay the same?)

$R, C = 1: \quad Q = \frac{1}{2}(\cos t + \sin t) + \cancel{ae^{-t}}$ ignore

To find the max we need to set $Q' = 0$ and solve for t:

$Q' = \frac{1}{2}(-\sin t + \cos t) = 0$

$-\sin t + \cos t = 0$

$\cos t = \sin t$ occurs at $t = \pi/4$

$\tan t = 1$

If R, C arbitrary, $\quad Q = \frac{1}{1+(RC)^2}\left(\cos t + RC^2 \sin t\right) + \cancel{ae^{-t/RC}}$ ignore

$Q' = 0 \Rightarrow \quad \frac{1}{1+(RC)^2}\left(-C\sin t + RC^2\cos t\right) = 0$

$RC^2 \cos t = C\sin t$

$\tan t = RC$

if RC large, then $t \to \pi/2$

if RC small, then $t \to 0$

(f) **Optional Challenge:** Use the identity

$$a\cos(t) + b\sin(t) = \sqrt{a^2+b^2}\cos\left(t - \arctan\left(\frac{b}{a}\right)\right)$$

To see if your last guess was correct.

From (e), $a = \frac{1}{2}, b = \frac{1}{2} \longrightarrow \quad \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\cos(t - \arctan 1)$

$\sqrt{\frac{1}{2}}\cos\left(t - \frac{\pi}{4}\right)$

max of cos occurs when $t - \pi/4 = 0 \Rightarrow t = \pi/4$, same as above