Math 33B Worksheet 2 (Exact ODEs)

Name: KEY Score: ______

Circle the name of your TA: Ziheng Nicholas Victoria

Circle the day of your discussion: Tuesday Thursday

1. **Warm Up:** Find the general (implicit) solution to

$$y^2 e^{xy} dx + (e^{xy} + xye^{xy} + 1)dy = 0$$

and solve for y if possible.

① $M = y^2 e^{xy}$ $N = e^{xy} + xye^{xy} + 1$

$M_y = 2ye^{xy} + xy^2 e^{xy} \xleftrightarrow{\text{same}} N_x = ye^{xy} + ye^{xy} + xy^2 e^{xy}$

② Want $\frac{\partial f}{\partial x} = M \Rightarrow f = \int M dx + g(y)$

$f = \int y^2 e^{xy} dx + g(y)$

$\underline{f = ye^{xy} + g(y)}$

③ Want $\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(ye^{xy} + g(y)\right) = e^{xy} + xye^{xy} + 1$

$\cancel{e^{xy}} + \cancel{xye^{xy}} + g'(y) = \cancel{e^{xy}} + \cancel{xye^{xy}} + 1$

④ $g'(y) = 1 \to g(y) = y$

⑤ Plug into ②: $f = ye^{xy} + y \to \boxed{ye^{xy} + y = C}$

* Check: $\frac{\partial f}{\partial x} = y^2 e^{xy} = M$ ✓

$\frac{\partial f}{\partial y} = e^{xy} + xye^{xy} + 1 = N$ ✓

2. **Integrating Factors:** Often times a differential form is not exact, but we can find a function μ so that

$$\mu P dx + \mu Q dy$$

is still exact. First, we consider what happens when μ depends only on a single variable:

(a) Assuming that $\mu(x, y) = \mu(y)$, i.e. that μ depends only on y, find a formula for μ'/μ in terms of P, Q, and their derivatives.

$M = \mu(y) P(x,y)$

$M_y = \mu'(y) P(x,y) + \mu(y) P_y(x,y)$

$N = \mu(y) Q(x,y)$

$N_x = \mu(y) Q_x(x,y)$

Want exact, i.e. $M_y = N_x$:

$\mu'(y) P + \mu(y) P_y = \mu(y) Q_x$

$\mu'(y) P = \mu(y)(Q_x - P_y)$

$\boxed{\frac{\mu'(y)}{\mu(y)} = \frac{Q_x - P_y}{P}}$

(b) Consider the form

$$(e^x + xe^x)\, dx + \left(\frac{xe^x}{y} + xe^x\right) dy = 0$$

Compute the function you found in part (a) for this example. Does it depend only on y? If so, we can find an integration factor that depends only on y. Find this integration factor.

Here $P = e^x(x+1)$ $Q = xe^x\left(\frac{1}{y} + 1\right)$

$P_y = 0$ $Q_x = (e^x + xe^x)\left(\frac{1}{y} + 1\right)$

$= e^x(1+x)\left(\frac{1}{y} + 1\right)$

not the same, so not exact

From (a), $\frac{\mu'(y)}{\mu} = \frac{\cancel{e^x(1+x)}\left(\frac{1}{y} + 1\right) - 0}{\cancel{e^x(x+1)}}$

$\frac{\mu'(y)}{\mu} = \frac{1}{y} + 1$

i.e. $\frac{1}{\mu}\frac{d\mu}{dy} = \frac{1}{y} + 1$, separable

$\frac{d\mu}{\mu} = \left(\frac{1}{y} + 1\right) dy$

$\int \frac{d\mu}{\mu} = \int \left(\frac{1}{y} + 1\right) dy$

$\ln|\mu| = \ln|y| + y$

$\mu = e^{\ln|y| + y}$

$= \boxed{ye^y}$

(c) Multiply equation by $\mu = ye^y$: $ye^y e^x(x+1)\,dx + ye^y xe^x(\frac{1}{y}+1)\,dy = 0$

$ye^{x+y}(x+1)\,dx + xe^{x+y}(1+y)\,dy = 0$

① $M = ye^{x+y}(x+1)$ $N = xe^{x+y}(1+y)$

$M_y = (e^{x+y}+ye^{x+y})(x+1)$ $N_x = (e^{x+y}+xe^{x+y})(1+y)$

$= (y+1)(x+1)e^{x+y} \xleftrightarrow{\text{Same!}} = (x+1)(y+1)e^{x+y}$

(c) Use the integration factor you found in the previous part to solve the ODE.

② Want $\frac{\partial f}{\partial x} = M$: $f = \int M\,dx + g(y)$

$f = \int (y+1)(x+1)e^{x+y}\,dx + g(y)$ parts $u = (y+1)(x+1)$, $dv = e^{x+y}dx$, $du = (y+1)dx$, $v = e^{x+y}$

$= (y+1)(x+1)e^{x+y} - \int (y+1)e^{x+y}dx + g(y)$

$= (y+1)(x+1)e^{x+y} - (y+1)e^{x+y} + g(y)$

$= (y+1)xe^{x+y} + g(y)$

③ Want $\frac{\partial f}{\partial y} = N$:

$\frac{\partial}{\partial y}\left((y+1)xe^{x+y} + g(y)\right) = xe^{x+y}(1+y)$

$xe^{x+y} + \cancel{(y+1)xe^{x+y}} + g'(y) = \cancel{xe^{x+y}(1+y)}$

④ $g'(y) = -xe^{x+y}$

$g(y) = -xe^{x+y}$

⑤ Plug into ②:

$f = (y+1)xe^{x+y} - xe^{x+y}$

$f = xye^{x+y}$

$\boxed{xye^{x+y} = C}$

* Check: $\frac{\partial f}{\partial x} = ye^{x+y} + xye^{x+y} = M$ ✓

$\frac{\partial f}{\partial y} = xe^{x+y} + xye^{x+y} = N$ ✓

A second example of integrating factor is when our form is homogeneous, i.e. that $P(tx, ty) = t^n P(x, y)$ and $Q(tx, ty) = t^n Q$ (note that this is unrelated to homogeneous linear ODEs). For concreteness, let's consider

$$\omega = -xy\,dx + (x^2 + y^2)dy$$

(d) We will use the change of variables $y = vx$; use the product rule to find dy in terms of dv and dx.

$y = vx$

$dy = x\,dv + v\,dx$ (product rule)

(e) Use this to find what the differential form is in terms of v and x. Plug in $y = vx$ and $dy = x\,dv + v\,dx$

$\omega = -x(vx)dx + (x^2 + (vx)^2)(x\,dv + v\,dx)$

$= \cancel{-vx^2dx} + x^3dv + \cancel{vx^2dx} + v^2x^3dv + v^3x^2dx$

$= v^3x^2dx + x^3(1+v^2)dv$

(f) Keeping in mind that separable forms (forms like $P(x)dx + Q(v)dv$) are exact, find a function you can divide by to turn this into a separable ODE.

Exact means $v^3x^2dx + x^3(1+v^2)\,dv = 0$ Separable equation

$v^3x^2dx = -x^3(1+v^2)dv$

$\frac{x^2}{x^3}dx = -\frac{(1+v^2)}{v^3}dv$ i.e. divide both sides by x^3v^3

$\frac{1}{x}dx = -\left(\frac{1}{v^3} + \frac{1}{v}\right)dv$

** Note since $y = vx$, this means we divide the original equation by y^3:

$-\frac{x}{y^2}dx + \frac{x^2+y^2}{y^3}dy = 0$

(g) Use this to solve the ODE in terms of v and x.

$\int \frac{1}{x}dx = -\int v^{-3} + \frac{1}{v}\,dv$

$\ln|x| = -\left(\frac{v^{-2}}{-2} + \ln|v|\right) + C$

$\ln|x| - \frac{1}{2v^2} + \ln|v| = C$

(h) Solve the ODE in terms of x and y (implicitly if necessary).

We used $y = vx \Rightarrow v = \frac{y}{x}$, so $\ln|x| - \frac{1}{2\left(\frac{y}{x}\right)^2} + \ln\left|\frac{y}{x}\right| = C$

$$\cancel{\ln|x|} - \frac{x^2}{2y^2} + \ln|y| - \cancel{\ln|x|} = C$$

$$\boxed{\ln|y| - \frac{x^2}{2y^2} = C}$$

Check: $\frac{\partial}{\partial x}\left(\ln|y| - \frac{x^2}{2y^2}\right) = -\frac{x}{y^2}$

$\frac{\partial}{\partial y}\left(\ln|y| - \frac{x^2}{2y^2}\right) = \frac{1}{y} - \frac{x^2}{\cancel{2}}(-\cancel{2}y^{-3}) = \frac{y^2}{y^3} + \frac{x^2}{y^3} = \frac{x^2+y^2}{y^3}$ } same as ** comment in (f)

3. **Potential Functions** A question we often ask in physics and mathematics is when a force is conservative, and if it is, what the corresponding potential energy is. Since we can interpret differential forms as vectors, this is a natural questions to ask of forms as well, and gives us another interpretation of what exact ODEs really are (they are also the most natural generalization of separable ODEs).

(a) What is the condition for a force to be conservative? (Think in terms of the multi-variable derivatives, gradient, divergence and curl).

If $\vec{F} = \langle P, Q \rangle$, need $\text{curl}(\vec{F}) = Q_x - P_y = 0 \Rightarrow P_y = Q_x$

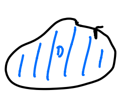
and F needs to be defined on a "simply connected region" (e.g. no holes)

(b) Show that this is equivalent to our condition for exactness.

The differential equation $P\,dx + Q\,dy = 0$ is exact if $P_y = Q_x$, same as (a)

(c) Use Stokes' theorem and line integrals to show that this condition implies a potential function exists (use line integrals to define the potential function, and show that it doesn't depend on the path taken using Stokes' theorem).

Stokes: $\int_C \vec{F}\cdot d\vec{r} = \int_D \text{curl}(\vec{F})\cdot d\vec{S}$



If $\text{curl}(\vec{F}) = 0$, then $\int_C \vec{F}\cdot d\vec{r} = 0$. But C is an arbitrary closed curve $\Rightarrow \vec{F}$ has a potential function f s.t. $\nabla f = \vec{F}$.

(d) The Gravitational and Electric forces both have the form

$$\frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}} = 0$$

find the corresponding potential function, and so solve the corresponding ODE.

① $M = \frac{x}{(x^2+y^2)^{3/2}} = x(x^2+y^2)^{-3/2}$ $\qquad N = \frac{y}{(x^2+y^2)^{3/2}} = y(x^2+y^2)^{-3/2}$

$M_y = -\frac{3}{2}x(x^2+y^2)^{-5/2}\cdot 2y \xleftrightarrow{\text{same}} N_x = -\frac{3}{2}y\cdot(x^2+y^2)^{-5/2}\cdot 2x \Rightarrow$ exact equation

② $\frac{\partial f}{\partial x} = M \Rightarrow f = \int M\,dx + g(y)$

$f = \int \frac{x}{(x^2+y^2)^{3/2}}\,dx + g(y) = \frac{1}{2}\int \frac{1}{u^{3/2}}\,du + g(y)$

$= \frac{1}{2}\left(\frac{-2}{1}\right)u^{-1/2} + g(y)$

$= \frac{-1}{(x^2+y^2)^{1/2}} + g(y)$

$u = x^2+y^2$

$du = 2x\,dx$

③ $\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(\frac{-1}{(x^2+y^2)^{1/2}} + g(y)\right) = \frac{y}{(x^2+y^2)^{3/2}}$

$-\frac{1}{2}\cdot\cancel{\frac{-1}{(x^2+y^2)^{3/2}}}\cdot 2y + g'(y) = \cancel{\frac{y}{(x^2+y^2)^{3/2}}}$

④ $g'(y) = 0$

$g(y) = C$

⑤ $f = \boxed{\frac{-1}{(x^2+y^2)^{1/2}} = C}$

* away from (0,0)

Check: $\frac{\partial f}{\partial x} = -1\cdot -\frac{1}{2}\,\frac{x}{(x^2+y^2)^{3/2}}$ ✓

$\frac{\partial f}{\partial y} = -1\cdot -\frac{1}{2}\,\frac{y}{(x^2+y^2)^{3/2}}$ ✓

4. **Challenge/Food for Thought:** In the previous example, the form field is not well defined at the origin. If we allow our forms to not exist at the origin, is the condition for having a potential function the same? Is

$$\frac{-ydx + xdy}{x^2 + y^2}$$

conservative?

$M = \frac{-y}{x^2+y^2}$ $\qquad$ $N = \frac{x}{x^2+y^2}$

$M_y = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$ $\qquad$ $N_x = \frac{(x^2+y^2)\cdot 1 - x(2x)}{x^2+y^2}$

$= \frac{-x^2+y^2}{(x^2+y^2)^2}$ $\xleftrightarrow{\text{same}}$ $= \frac{-x^2+y^2}{(x^2+y^2)^2}$

Domain is not simply connected since undefined at (0,0) (a hole) $\Rightarrow$ NOT conservative