Math 33B Worksheet 3 (Equlibria and Exact ODEs)

Name: KEY Score: ____________

Circle the name of your TA: Ziheng Nicholas Victoria

Circle the day of your discussion: Tuesday Thursday

1. To begin with, let's return to an example from biology. If we have a population, P, of animals, we can speculate that, if there is a finite amount of resources, the population might develop as

$$P' = rP(C - P)$$

where C is called the carrying capacity of the environment.

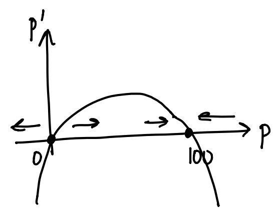
(a) Take C=100, and draw the phase line of the system, and use it to identify equilibria of the system.

$P' = rP(100-P)$

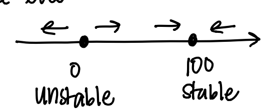
$rP(100-P) = 0$

$\boxed{P=0 \text{ or } P=100}$

are equilibria



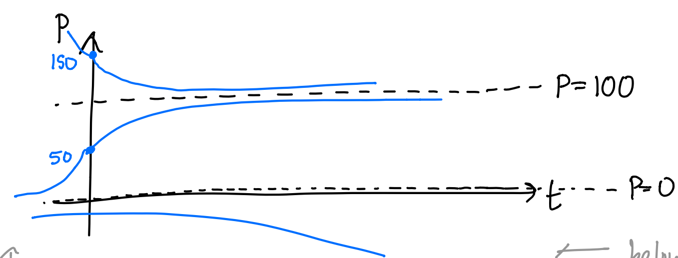
Phase line:



(b) Classify the equilibria as stable or unstable. Use your classification to draw a plot of various solutions. Include the equilibrium solutions in your plot, as well solutions passing through y=50 and y=150. ← I am interpreting this to mean $P_0 = 50$, $P_0 = 150$

$P=0$ is unstable

$P=100$ is stable



left of P axis means time is negative

← below x-axis isn't plausible because cannot have negative initial population

(c) Assuming we start with a positive population, what does our model predicate the size of our population will be after a long time? If we start off with 50 animals, can we ever have 101? How do you know?

As $t \to \infty$, $P \to 100$

If $P_0 = 50$, cannot reach 101

2. Sometimes we can solve autonomous systems exactly, but does this make it easier to understand the properties of the solutions? Consider

$$x' = sin(x)$$

(a) Find the general solution to this ODE.

$\frac{dx}{dt} = \sin x$ separable equation

$\frac{dx}{\sin x} = dt$

$\int \csc x \, dx = \int dt$

$\boxed{-\ln|\csc x + \cot x| = t + C}$

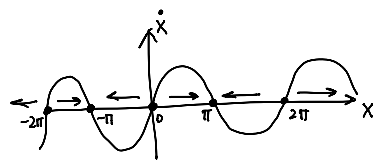
(b) Find the equilibria of the system, and classify their stability.

$x' = \sin x$

$\sin x = 0$

$x = 0, \pm\pi, \pm 2\pi, \ldots$

$= n\pi$ where n is integer



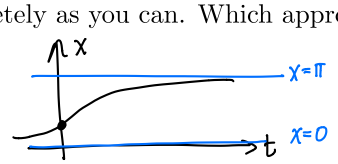
$x = 2n\pi$, n integer (even multiple of π)

$\Rightarrow$ unstable

$x = (2n+1)\pi$, n integer (odd multiple of π)

$\Rightarrow$ stable

(c) Using either part a or part b, determine the large time limit of the IVP $x(0) = 1$. Also determine whether any solution can have large time limit of ∞. Justify your answer as completely as you can. Which approach seems easier to you?



If $x(0)=1$ then $x \to \pi$ as $t \to \infty$

(using (b) is easier)

3. **Basic Bifurcation Theory** Consider two new systems, P and Q, which satisfy the ODEs

$$P' = P(C - P)$$

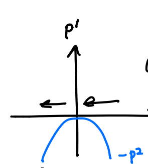
$$Q' = C + Q^2$$

Determine the equilibria of each system, and classify their stability. Then draw of plot of the value of the equilibria as a function of C, keeping track of which equilibria are stable and which are not. (Notice that small changes in the ODE can result in significant changes in the number or stability of equilibria.)

$P' = P(C-P)$

3 cases: (i) $C=0$: $P' = -P^2$

$-P^2 = 0 \Rightarrow P = 0$



derivative decreasing

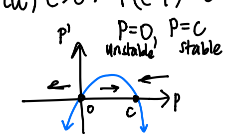
Check solution: $\int \frac{dP}{-P^2} = \int dt \Rightarrow -\frac{1}{P} = t + C \Rightarrow P = \frac{-1}{t+C}$

as $t \to \infty$, $P \to 0$, $P=0$ is stable

(ii) $C>0$: $P(C-P)=0$

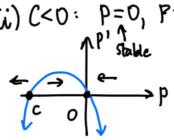
$P=0$, $P=C$

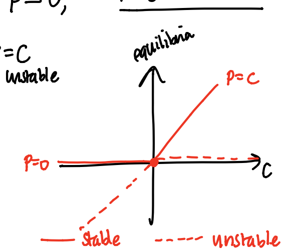
unstable, stable



(iii) $C<0$: $P=0$, $P=C$

stable, unstable

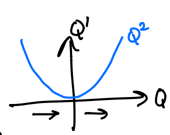




$Q' = C + Q^2$

(i) $C=0$: $Q^2 = 0$

$Q = 0$



check solution: $\int \frac{dQ}{Q^2} = \int dt$ $\quad -\frac{1}{Q} = t + C$

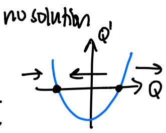
$Q = \frac{-1}{t+C} \to 0$ as $t \to \infty \Rightarrow Q = 0$ is stable

(ii) $C>0$: $C + Q^2 = 0$ no solution

(iii) $C<0$: $C + Q^2 = 0$

$Q^2 = -C$

$Q = \pm\sqrt{-C}$



$Q = -\sqrt{-C}$ stable

$Q = \sqrt{-C}$ unstable