Eigenvalues, Eigenvectors, Diagonalization

November 24, 2015

We say that λ is an **eigenvalue** with nonzero **eigenvector** $\mathbf{v}$ of a matrix A iff $A\mathbf{v} = \lambda\mathbf{v}$.

To solve for eigenvalues and eigenvectors, we look at the definition $A\mathbf{v} = \lambda\mathbf{v}$. Rewrite the right hand side as $\lambda I\mathbf{v}$:

$$A\mathbf{v} = \lambda I\mathbf{v}$$

Subtract everything over to one side:

$$A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0}$$

Factor out $\mathbf{v}$ from the right hand side:

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{v}$ is nonzero, it is a nontrivial solution to the matrix equation. To have a nontrivial solution means that matrix $A - \lambda I$ is not invertible, or that $\det(A - \lambda I) = 0$. Thus to solve for eigenvalues, we solve the equation $\det(A - \lambda I) = 0$ for λ. To solve for the corresponding eigenvectors, we solve the matrix equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Example. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$.

Solution. Step 1: Find the eigenvalues of A. We calculate $\det(A - \lambda I)$:

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - (-1)(0) = (1-\lambda)(2-\lambda).$$

We call $p(\lambda) = (1-\lambda)(2-\lambda)$ the **characteristic polynomial**. We set $\det(A - \lambda I) = p(\lambda) = 0$ and solve for λ:

$$(1-\lambda)(2-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 1 \text{ mult } 1, \lambda_2 = 2 \text{ mult } 1.$$

Note: We say that these eigenvalues have multiplicity 1 because it only occured once. In other words, look at the exponent. For example, if we had the polynomial $(1-\lambda)^3(2-\lambda)^5$ then the eigenvalues would be $\lambda_1 = 1$ mult. 3 and $\lambda_2 = 2$ mult. 5. The multiplicity tells us how many eigenvectors are associated with that eigenvalue. In this example, our eigenvalues each have one eigenvector.

Notice that $\lambda_1 = 1$ and $\lambda_2 = 2$ are actually the diagonal entries of A. The diagonal entries of a diagonal or triangular matrix are the eigenvalues.

Step 2: Find the eigenvectors of A. We need to plug λ_1 and λ_2 into the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$. When $\lambda_1 = 1$:

$$\left(\begin{array}{cc|c} 1-\lambda_1 & 0 & 0 \\ -1 & 2-\lambda_1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 1 & 0 \end{array}\right).$$

x_2 is a free variable, and the second equation tells us that $x_1 = x_2$. Therefore the general solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2,$$

and the basis of the nullspace of $A-\lambda_1 I$ is $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$. This implies that the eigenvector corresponding with λ_1 is given by $\mathbf{v}_{\lambda_1} = \begin{pmatrix}1\\1\end{pmatrix}$.

Now repeat for the next eigenvalue. When $\lambda_2 = 2$:

$$\left(\begin{array}{cc|c} 1-\lambda_2 & 0 & 0 \\ -1 & 2-\lambda_2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) \xrightarrow{-R1+R2\to R2} \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

x_2 is a free variable, and the first equation tells us that $x_1 = 0$. Therefore the general solution is given by

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\x_2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} x_2,$$

and the basis of the nullspace of $A-\lambda_2 I$ is $\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$. This implies that the eigenvector corresponding with λ_2 is given by $\mathbf{v}_{\lambda_2} = \begin{pmatrix}0\\1\end{pmatrix}$.

To summarize, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$ with eigenvectors $\mathbf{v}_{\lambda_1} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v}_{\lambda_2} = \begin{pmatrix}0\\1\end{pmatrix}$. □

Sometimes we wish to find powers of matrices, like A^{100}. But it would be a very difficult and tedious process to multiply a matrix A out 100 times. To make this process easier we **diagonalize** a matrix: If A is an $n \times n$ matrix and has n distinct eigenvectors, then $A = PDP^{-1}$ where P is the matrix of eigenvectors of A:

$$P = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n\end{pmatrix}$$

and D is a diagonal matrix of eigenvalues of A:

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{pmatrix}.$$

If $A = PDP^{-1}$ then

$$\begin{aligned} A^k &= A \cdot A \cdots A \\ &= PDP^{-1} \cdot PDP^{-1} \cdots PDP^{-1} \\ &= PD(P^{-1}P)D(P^{-1}P) \cdots (P^{-1}P)DP^{-1} \\ &= PDD \cdots DP^{-1} \\ &= PD^kP^{-1}. \end{aligned}$$

Example. Find A^{10} where $A = \begin{pmatrix}1 & 0\\-1 & 2\end{pmatrix}$.

Solution. Our goal is to diagonalize A. We continue the steps from the previous example. We found that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$ with eigenvectors of A are $\mathbf{v}_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_{\lambda_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Step 3: Form the matrix P of eigenvectors and the diagonal matrix D of eigenvalues. Order matters! If we use $\mathbf{v}_{\lambda_1}$ in the first column of P then we must also use λ_1 in the first column of D:

$$P = \begin{pmatrix} \mathbf{v}_{\lambda_1} & \mathbf{v}_{\lambda_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

If we were to use $\mathbf{v}_{\lambda_2}$ in the first column of P then we must also use λ_2 in the first column of D; this would look like:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Just stay consistent with your steps. For now we will use the P and D that had $\mathbf{v}_{\lambda_1}$ and λ_1 in the first column.

Step 4: Find P^{-1}. P is a 2×2 matrix and so P^{-1} is easy to find:

$$P^{-1} = \frac{1}{1 \cdot 1 - 0 \cdot 1} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Step 5: Show that $A = PDP^{-1}$.

$$\begin{aligned} PDP^{-1} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \\ &= A. \end{aligned}$$

Step 6: Find A^{10}. Now that we have diagonalized A,

$$\begin{aligned}
A^{10} &= PD^{10}P^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1024 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -1023 & 1024 \end{pmatrix}.
\end{aligned}$$

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