Ex Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Eigenvalues: $|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix}$

$\det(A-\lambda I)=0$

first row

$$= -\lambda\left[(2-\lambda)(3-\lambda)-0\right] - 0 \; -2\left(0-(2-\lambda)\right)$$

$$= -\lambda(2-\lambda)(3-\lambda) + 2(2-\lambda)$$

$$= (2-\lambda)\left[-\lambda(3-\lambda)+2\right] = (2-\lambda)\left[\lambda^2-3\lambda+2\right]$$

$$= (2-\lambda)(\lambda-2)(\lambda-1) = 0$$

$\boxed{\lambda_1 = 2 \text{ mult } 2, \quad \lambda_2 = 1 \text{ mult } 1}$

Eigenvectors $(A-\lambda I)\vec{v} = \vec{0}$

$\lambda_1 = 2$: $\left(\begin{array}{ccc|c} -\lambda_1 & 0 & -2 & 0 \\ 1 & 2-\lambda_1 & 1 & 0 \\ 1 & 0 & 3-\lambda_1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

two free variables x_2, x_3

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$

$\boxed{\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$\lambda_2 = 1$: $\left(\begin{array}{ccc|c} -\lambda_2 & 0 & -2 & 0 \\ 1 & 2-\lambda_2 & 1 & 0 \\ 1 & 0 & 3-\lambda_1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \cancel{\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

One free variable x_3

$x_1 + 2x_3 = 0 \qquad x_1 = -2x_3$

$x_2 - x_3 = 0 \qquad x_2 = x_3$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} x_3$

$\boxed{\vec{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}$

Some facts: • If A is a diagonal or upper/lower Δ matrix, the eigenvalues are its diagonal entries (consequence of determinant properties)

• If A has eigenvalue $\lambda = 0$, then it is not invertible.

Diagonalization

Sometimes we want to find powers of matrices like A^{100}, but this is difficult. To make the calculation easier, we will <u>diagonalize</u> the matrix:

$$A = P D P^{-1}$$

where $P = (\vec{v_1}\ \vec{v_2}\ \cdots\ \vec{v_n})$ and $D = \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \lambda_n \end{pmatrix}}_{\text{diagonal matrix}}$

$\vec{v_1}, \dots, \vec{v_n}$ eigenvectors, $\lambda_1, \dots, \lambda_n$ eigenvalues.

Then $A^k = (PDP^{-1})^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$

$$= P \underbrace{D \cdots D}_{k \text{ times}} P^{-1}$$

$$= P D^k P^{-1},$$

and $D^k = \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}.$

<u>Ex</u> Find A^{13} where $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$

We found eigenvalues and eigenvectors in previous example.

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad P = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Ex continued

need to find P^{-1}.

$$\left(\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & +1 & -1 & 0 & -1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

Then $A = PDP^{-1} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$.

and $A^{13} = PD^{13}P^{-1} = \underbrace{\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1 \end{pmatrix}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -2^{13} & -2 \\ 2^{13} & 0 & 1 \\ 0 & 2^{13} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} -2^{13}+2 & 0 & -2^{14}+2 \\ 2^{13}-1 & 2^{13} & 2^{13}-1 \\ 2^{13}-1 & 0 & 2^{14}-1 \end{pmatrix}}$$

Ex Find A^{10} where $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

eigenvalues: $\boxed{\lambda_1 = 1, \quad \lambda_2 = 2}$ ← fast because lower △ matrix

eigenvectors: $\lambda_1 = 1$: $\left(\begin{array}{cc|c} 1-1 & 0 & 0 \\ -1 & 2-1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & +1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ x_2 free, $x_1 - x_2 = 0$, $x_1 = x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2 \qquad \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\lambda_2 = 2$: $\left(\begin{array}{cc|c} 1-2 & 0 & 0 \\ -1 & 2-2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ x_2 free, $x_1 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2 \qquad \boxed{\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\qquad P^{-1} = \frac{1}{1-0}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

Ex continued

$$A = PDP^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$A^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1024 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1-1024 & 1024 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \\ -1023 & 1024 \end{pmatrix}}$$

Another way to find eigenvalues:

characteristic polynomial of A, 2×2 is given by $p(\lambda) = \lambda^2 - tr(A)\lambda + \det A$

set equal to 0 and solve for λ.

Complex Eigenvalues and Eigenvectors

Same steps as before, just note that complex eigenvalues and eigenvectors come in "pairs."
(what we call "complex conjugates")

Ex Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$

eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda) + 5 = 24 - 10\lambda + \lambda^2 + 5$

$= \lambda^2 - 10\lambda + 29$ cannot factor $\therefore$ use quadratic formula

$$\lambda = \frac{10 \pm \sqrt{100 - 4(1)(29)}}{2} = \frac{10 \pm \sqrt{-16}}{2} = \frac{10 \pm 4i}{2} = 5 \pm 2i$$

$\boxed{\lambda_1 = 5+2i \qquad \lambda_2 = 5-2i}$ ← they are complex conjugates of each other!

eigenvectors: $\lambda_1 = 5+2i$ $\left(\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right)$

multiply the first row by $-(1+2i)$ and add to second row

i.e. $-(1+2i)R_1 + R_2 \to R_2$

$$1 - \cancel{2i} + \cancel{2i} - 4i^2$$

$$\begin{array}{ccc} -(1+2i)(1-2i) & -(1+2i)(-1) & 0 \\ 5 & -1-2i & 0 \\ \hline \end{array} \rightarrow \begin{array}{ccc} -5 & 1+2i & 0 \\ 5 & -1-2i & 0 \\ \hline 0 & 0 & 0 \end{array}$$

$$\left(\begin{array}{cc|c} 1-2i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

one free variable

$(1-2i)x_1 - x_2 = 0 \rightarrow (1-2i)x_1 = x_2$ let's say x_1 free

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ (1-2i)x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} x_1 \qquad \boxed{\vec{v_1} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}}$$

Because eigenvectors come in pairs, then $\boxed{\vec{v_2} = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}}$