Case (iii) Repeated eigenvectors (2x2 case)

If λ is an eigenvalue with eigenvector $\vec{v}_1$, then

$$\vec{X} = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}(\vec{v}_1 t + \vec{v}_2)$$

where $(A-\lambda I)\vec{v}_2 = \vec{v}_1$ ($\vec{v}_2$ is called the generalized eigenvector)

Ex Solve the system $\vec{X}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\vec{X}$

① eigenvalues $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (3-\lambda)(-9-\lambda)+36$

$-27-3\lambda+9\lambda+\lambda^2+36 = 0 \Rightarrow \lambda^2+6\lambda+9=0 \Rightarrow (\lambda+3)^2 = 0$

$\lambda = -3$ mult 2

② eigenvectors

$(A-\lambda I)\vec{v}_1 = 0$: $\left(\begin{array}{cc|c} 3-(-3) & -18 & 0 \\ 2 & -9-(-3) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right)$ $2x_1 = 6x_2$, $x_1 = 3x_2$ $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$(A-\lambda I)\vec{v}_2 = \vec{v}_1$ $\left(\begin{array}{cc|c} 3-(-3) & -18 & 3 \\ 2 & -9+3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 1 \end{array}\right)$ $2x_1 = 6x_2+1$, $x_1 = 3x_2 + \frac{1}{2}$, $x_2 = 0$ $\vec{v}_2 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

$$\boxed{\vec{X} = c_1 e^{-3t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-3t}\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right]}$$

Nonhomogeneous Linear Systems – Undetermined Coefficients

Want to solve equations of the form $\vec{x}' = A\vec{x} + \vec{g}(t)$.

where $\vec{g}(t)$ is a vector (or vectors) of exponential functions, polynomials, sine/cosine functions or any sum or product of these functions.

<u>Steps to Solve</u>

1. Write in Standard form: $\vec{x}' = A\vec{x} + \vec{g}(t)$
2. Solve for homogeneous solution $\vec{x}_h$ $\quad (\vec{x}' = A\vec{x})$
3. Guess $\vec{x}_p$
4. Plug $\vec{x}_p$ in and solve for coefficients
5. $\vec{x} = \vec{x}_p + \vec{x}_h$

How to guess? Similar to before...

$\vec{g}(t)$	$\vec{x}_p$
$\begin{pmatrix}100\\0\end{pmatrix}e^t$	$\vec{a}e^t = \begin{pmatrix}a_1\\a_2\end{pmatrix}e^t$
$\begin{pmatrix}18\\-2\end{pmatrix}t^2 - \begin{pmatrix}4\\3\end{pmatrix}t$	$\vec{a}t^2 + \vec{b}t + \vec{c} = \begin{pmatrix}a_1\\a_2\end{pmatrix}t^2 + \begin{pmatrix}b_1\\b_2\end{pmatrix}t + \begin{pmatrix}c_1\\c_2\end{pmatrix}$
[illegible] $+ \begin{pmatrix}3\\1\end{pmatrix}\cos 2t$	$\vec{a}\sin 2t + \vec{b}\cos 2t = \begin{pmatrix}a_1\\a_2\end{pmatrix}\sin 2t + \begin{pmatrix}b_1\\b_2\end{pmatrix}\cos 2t$

What happens if $\vec{g}(t)$ has functions in common with $\vec{x}_h$?
a little different than before

<u>Ex</u> Suppose $\vec{x}_h = c_1\begin{pmatrix}1\\2\end{pmatrix}e^t + c_2\begin{pmatrix}3\\4\end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix}9\\4\end{pmatrix}$. Find form of $\vec{x}_p$.

Guess: $\vec{x}_p = \cancel{\overset{\vec{a}}{\begin{pmatrix}a_1\\a_2\end{pmatrix}}}$ but already shows up in $\vec{x}_h$! $\left(c_2\begin{pmatrix}3\\4\end{pmatrix}\right)$

$$\vec{x}_p = \vec{a}t + \vec{b} = \begin{pmatrix}a_1\\a_2\end{pmatrix}t + \begin{pmatrix}b_1\\b_2\end{pmatrix}$$

<u>Ex</u> Suppose $\vec{x}_h = c_1\begin{pmatrix}1\\2\end{pmatrix}e^t + c_2\begin{pmatrix}3\\4\end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix}-3\\1\end{pmatrix}e^t + \begin{pmatrix}9\\4\end{pmatrix}$. Find form of $\vec{x}_p$.

Guess: $\vec{x} = \cancel{\vec{a}e^t + \vec{b}} = \underline{\underline{\begin{pmatrix}a_1\\a_2\end{pmatrix}e^t}} + \underline{\underline{\begin{pmatrix}b_1\\b_2\end{pmatrix}}}$ already part of $\vec{x}_h$

$$\vec{x} = \vec{a}te^t + \vec{b}e^t + \vec{c}t + \vec{d} = \begin{pmatrix}a_1\\a_2\end{pmatrix}te^t + \begin{pmatrix}b_1\\b_2\end{pmatrix}e^t + \begin{pmatrix}c_1\\c_2\end{pmatrix}t + \begin{pmatrix}d_1\\d_2\end{pmatrix}$$

<u>Ex</u> Solve $\begin{cases} \dfrac{dx}{dt} = 5x + 9y + 2 \\ \dfrac{dy}{dt} = -x + 11y + 6 \end{cases}$

① standard form $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ -1 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$$\vec{x}\,' = \begin{pmatrix} 5 & 9 \\ -1 & 11 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

② Solve $\vec{x}\,' = \begin{pmatrix} 5 & 9 \\ -1 & 11 \end{pmatrix} \vec{x}$.

$$\begin{vmatrix} 5-\lambda & 9 \\ -1 & 11-\lambda \end{vmatrix} = (5-\lambda)(11-\lambda) + 9 = 55 - 16\lambda + \lambda^2 + 9 = \lambda^2 - 16\lambda + 64$$

$$= (\lambda - 8)^2 = 0 \Rightarrow \lambda = 8 \text{ mult } 2$$

$$\left(\begin{array}{cc|c} 5-8 & 9 & 0 \\ -1 & 11-8 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -3 & 9 & 0 \\ -1 & 3 & 0 \end{array}\right) \quad \begin{array}{l} -x_1 + 3x_2 = 0 \\ x_1 = 3x_2 \end{array} \quad x_1 = 0 \quad \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -3 & 9 & 3 \\ -1 & 3 & 1 \end{array}\right) \quad \begin{array}{l} -x_1 + 3x_2 = 1 \\ x_1 = 3x_2 - 1 \end{array} \quad x_2 = 0 \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\vec{x}_h = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + c_2 e^{8t} \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]$$

③ $\vec{g}(t) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ $\qquad \vec{x}_p = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

④ $\vec{x}_p\,' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ -1 & 11 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5a_1 + 9a_2 + 2 \\ -a_1 + 11a_2 + 6 \end{pmatrix}$$

$$\begin{cases} 5a_1 + 9a_2 = -2 \\ -a_1 + 11a_2 = -6 \end{cases} \Rightarrow \begin{array}{r} \cancel{5a_1} + 9a_2 = -2 \\ -\cancel{5a_1} + 55a_2 = -30 \\ \hline 64a_2 = -32 \end{array} \quad a_2 = -1/2$$

$$a_1 = 11a_2 + 6$$

$$a_1 = -\frac{11}{2} + 6 = \frac{1}{2} \quad \Rightarrow \quad \vec{x}_p = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

⑤ $\vec{x} = \vec{x}_h + \vec{x}_p \Rightarrow \boxed{\vec{x} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + c_2 e^{8t} \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] + \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}}$

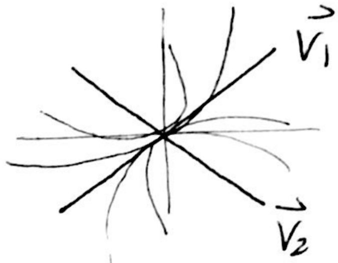
Stability and Classification $\quad \vec{x}' = A\vec{x} \quad A \; 2\times 2$

If $\lim\limits_{t\to\infty} \vec{x}$ converges, we say that the solution is <u>stable</u>.

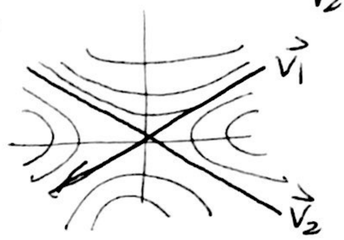
Otherwise it is <u>unstable</u>.

Classification: (i) Real, distinct eigenvalues λ_1, λ_2

- λ_1, λ_2 same sign : node
  - $+$ unstable
  - $-$ stable

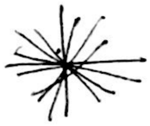


- λ_1, λ_2 opposite sign : saddle
  - always unstable

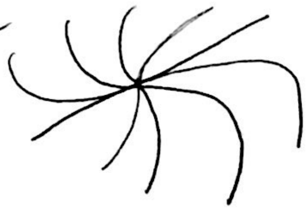


(ii) Real, repeated eigenvalue $\lambda \quad$ $+$ unstable, $-$ stable

- two L.I eigenvectors : star

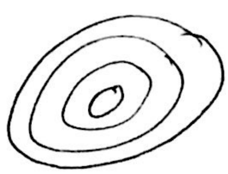


- one eigenvector : degenerate node



(iii) complex eigenvalues

- real part $= 0 \quad$ center (stable)

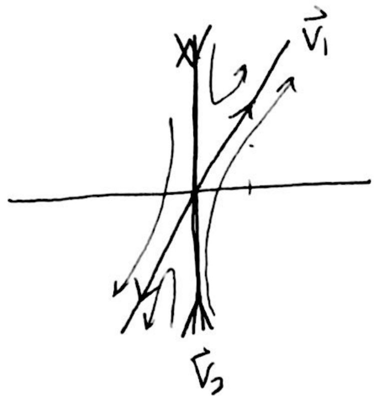


- real part $\neq 0 \quad$ spiral
  - real part $+ \quad$ unstable
  - real part $- \quad$ stable

Previous examples...

Ex (a) $\vec{x}' = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\vec{x}$ has solution $\vec{x} = c_1 e^{3t}\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{v}_1} + c_2 e^{-t}\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{v}_2}$

$\lambda_1 = 3$, $\lambda_2 = -1$ opposite signs $\Rightarrow$ saddle, unstable



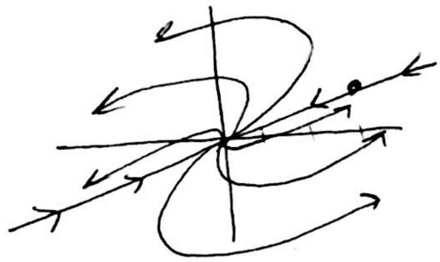
plot $\vec{v}_1$ $\vec{v}_2$

$\vec{v}_1$ outward arrow (diverges)

$\vec{v}_2$ inward arrows (converges

(b) $\vec{x}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\vec{x}$ has solution $\vec{x} = c_1 e^{-3t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-3t}\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix}t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right]$

$\lambda = -3$ stable, degenerate node



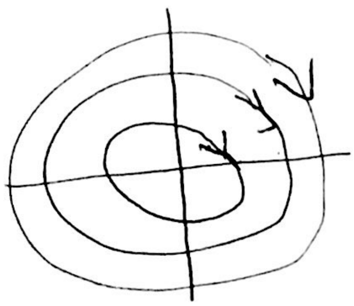
plot $\vec{v}_1$ (inward arrow)

direction: look at tangent to a point

at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\vec{x}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ upward

(c) $\vec{x}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\vec{x}$ has solution $\vec{x} = c_1\begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \end{pmatrix} + c_2\begin{pmatrix} -2\sin 2t - 2\cos 2t \\ \sin 2t \end{pmatrix}$

$\lambda = \pm 2i$ real part $0 \Rightarrow$ center, stable



find tangent at point for direction:

$\vec{x}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ downward

clockwise