Example Using d'Alembert's Method

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Use d'Alembert's method to solve the partial differential equation system

$$u_{tt} = \frac{1}{\pi^2} u_{xx}, u(x,0) = \sin \pi x, u_t(x,0) = 0, -\infty < x < \infty, t \geq 0.$$

Solution. We are given $c = \frac{1}{\pi}$, $f(x) = \sin \pi x$, $g(x) = 0$. The solution to the PDE is then

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f\left(x - \frac{1}{\pi}t\right) + f\left(x + \frac{1}{\pi}t\right)\right] + \frac{\pi}{2}\int_{x-t/\pi}^{x+t/\pi} g(s)ds \\ &= \frac{1}{2}\left[\sin(\pi x - t) + \sin(\pi x + t)\right] + \frac{\pi}{2}\int_{x-t/\pi}^{x+t/\pi} 0ds \end{aligned}$$

Use the identity $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ to simplify:

$$\begin{aligned} \sin(\pi x - t) &= \sin \pi x \cos t - \sin t \cos \pi x \\ \sin(\pi x + t) &= \sin \pi x \cos t + \sin t \cos \pi x. \end{aligned}$$

Plug these into our general solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(\pi x - t) + \sin(\pi x + t)\right] \\ &= \frac{1}{2}\left[\sin \pi x \cos t - \sin t \cos \pi x + \sin \pi x \cos t + \sin t \cos \pi x\right] \\ &= \sin \pi x \cos t. \end{aligned}$$

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