Math 6B Practice Problems II

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1. (a) $R = \infty, I = (-\infty, \infty)$ (b) $R = 4,\ I = (-4, 4]$ (c) $R = 0, I = \{\frac{1}{2}\}$

2. (a) $f(x) = -\sum_{n=0}^{\infty} x^{7n}$ (b) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{2^{n+1}}$

3. $-\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x+3)^n$

4. (a) $\sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} x^{2n+1} + C$

   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(6n+2)(2n)!} + C$

5. $\frac{1}{3}$

6. (a) e^{x^4}

   (b) $-1 + e^3$

7. (a) $f(x) = \frac{3}{2} - \sin x + 3\cos 2x - \frac{1}{2}\cos 6x$

   (b) $f(x) = \frac{p^2}{3} + \sum_{n=0}^{\infty} \frac{4p^2}{(n\pi)^2}(-1)^n \cos\left(\frac{n\pi}{p}x\right)$

   (c) $f(x) = \sum_{n=1}^{\infty} \frac{16}{\pi}(-1)^{n+1} \frac{n}{(2n+1)^2(2n-1)^2} \sin(2nx)$

8. Cosine series: $\frac{1}{2} + \sum_{k=0}^{\infty} \frac{-4}{(2k+1)^2\pi^2} \cos(2k+1)x$

   Sine series: $\sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin(n\pi x)$

9. (a) $\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$

   (b) $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1 - i\omega}{1 + \omega^2}$

   (c) $\hat{f}(\omega) = -\sqrt{\frac{2}{\pi}} \frac{2(i\omega + 1)\sin \omega}{\omega^3}$

10. $u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin(n\pi x)\cos(4n\pi t)$

11. $u(x,t) = \sin \pi x \cos \pi t + 3 \sin 2\pi x \cos 2\pi t + \frac{1}{\pi} \sin \pi x \sin \pi t$

12. $u(x,t) = \sum_{n=1}^{\infty} -\frac{2n\pi}{(n\pi)^2 + 1} \left(\frac{(-1)^n}{e} + 1\right) e^{-(n\pi)^2 t} \sin(n\pi x)$

Detailed Solutions

1. Find the radius and interval convergence of the series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

   Solution. This is a power series about $a = 0$. Using the Ratio Test with $a_n = \frac{x^n}{n!}$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1$$

   Since our limit is always less than 1, the radius of convergence is $R = \infty$ and the interval of convergence is $I = (-\infty, \infty)$. $\square$

   (b) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{4^n \ln n}$

   Solution. This is a power series about $a = 0$. Using the Ratio test with $a_n = (-1)^n\frac{x^n}{4^n \ln n}$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{4^{n+1}\ln(n+1)}\cdot\frac{4^n\ln n}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{|x|}{4}\frac{\ln n}{\ln(n+1)} \overset{L'H}{=} \lim_{n\to\infty}\frac{|x|}{4}\frac{n+1}{n} = \frac{|x|}{4} < 1$$

   "L'H" denotes where we used L'Hôpital's Rule to evaluate the limit. We have $|x| < 4$, hence the radius of convergence is $R = 4$. We need to test the endpoints at $x = -4, x = 4$.
   When $x = -4$:

$$\sum_{n=2}^{\infty}(-1)^n\frac{(-4)^n}{4^n\ln n} = \sum_{n=2}^{\infty}\frac{1}{\ln n}.$$

   Since $n > \ln n$ (graph it), then $\frac{1}{\ln n} > \frac{1}{n}$. The sum $\sum\frac{1}{n}$ diverges, therefore by the Comparison Test, $\sum\frac{1}{\ln n}$ must also diverge. We do not include $x = -4$ in the interval of convergence.
   When $x = 4$:

$$\sum_{n=2}^{\infty}(-1)^n\frac{(-4)^n}{4^n\ln n} \quad \sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}.$$

   The sequence $b_n = \frac{1}{\ln n}$ is decreasing and $\lim_{n\to\infty} b_n = 0$. Thus by the Alternating Series test, the series $\sum\frac{(-1)^n}{\ln n}$ converges. We include $x = 4$ in the interval of convergence.
   Therefore the interval of convergence is $(-4, 4]$. $\square$

   (c) $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

   Solution. This is a power series about $a = \frac{1}{2}$ since $(2x-1)^n = [2(x-\frac{1}{2})]^n$. Using the ratio test with $a_n = n!(2x-1)^n$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)|2x-1| = \infty.$$

   The limit is never less than 1, so the radius of convergence is $R = 0$. Therefore the interval of convergence is just the point $\{\frac{1}{2}\}$. $\square$

2. Use the power series $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ to find the power series of the following functions:

(a) $f(x) = \dfrac{1}{x^7 - 1}$

Solution.

$$f(x) = -\frac{1}{1-x^7} = -\sum_{n=0}^{\infty}(x^7)^n = -\sum_{n=0}^{\infty} x^{7n}.$$

□

(b) $f(x) = \dfrac{x^3}{x+2}$

Solution.

$$f(x) = \frac{x^3}{2} \cdot \frac{1}{1+\frac{x}{2}} = \frac{x^3}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{x^3}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+3}}{2^{n+1}}.$$

□

3. Find the Taylor series for $f(x) = \dfrac{1}{x}$ centered at $a = -3$.

Solution. The Taylor series of $f(x)$ about $a = -3$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Find a pattern for the nth derivative:

$$\begin{aligned}
f(x) &= x^{-1} \\
f'(x) &= (-1)x^{-2} \\
f''(x) &= (-1)(-2)x^{-3} \\
f'''(x) &= (-1)(-2)(-3)x^{-4} \\
&\vdots \\
f^{(n)}(x) &= (-1)(-2)\cdots(-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}
\end{aligned}$$

Evaluate the nth derivative at $a = -3$:

$$f^{(n)}(-3) = (-1)^n n!(-3)^{-(n+1)} = \frac{(-1)^n n!}{(-1)^{n+1}3^{n+1}} = \frac{(-1)n!}{3^{n+1}}.$$

Therefore the Taylor series about $a = -3$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{(-1)n!}{3^{n+1}n!}(x+3)^n = -\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x+3)^n.$$

□

4. Evaluate the indefinite integral as an infinite series:

(a) $\int e^{x^2} dx$

Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Then

$$\begin{aligned} \int e^{x^2} dx &= \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} dx = \int \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + ...\right) dx = x + \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + ... + C \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} x^{2n+1} + C \end{aligned}$$

□

(b) $\int x \cos(x^3) dx$

Solution. Since $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$,

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}.$$

Then

$$\begin{aligned} \int x \cos(x^3) dx &= \int x \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} dx = \int x \left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + ...\right) dx \\ &= \int \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + ...\right) dx = \frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} + ... + C \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(6n+2)(2n)!} + C \end{aligned}$$

□

5. Use series to evaluate the limit $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$.

Solution. Since $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, then

$$\begin{aligned} \lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} &= \lim_{x \to 0} \frac{x - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x^3} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)}{x^3} \\ &= \lim_{x \to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5!} + \frac{x^7}{7!} + ...}{x^3} = \lim_{x \to 0} \frac{1}{3} - \frac{x^2}{5!} + \frac{x^4}{7!} + ... = \frac{1}{3}. \end{aligned}$$

□

6. Find the sum of the series:

   (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}$

   Solution. Rewrite the series as
$$\sum_{n=0}^{\infty}\frac{(-x^4)^n}{n!}=e^{-x^4}.$$
$\square$

   (b) $3+\dfrac{9}{2!}+\dfrac{27}{3!}+\dfrac{81}{4!}+...$

   Solution. We can write the sum as the series
$$\sum_{n=1}^{\infty}\frac{3^n}{n!}=-\frac{3^0}{0!}+\frac{3^0}{0!}+\sum_{n=1}^{\infty}\frac{3^n}{n!}=-1+\sum_{n=0}^{\infty}\frac{3^n}{n!}=-1+e^3.$$
$\square$

7. Find the Fourier series of the function:

   (a) $f(x)=1-\sin x+3\cos 2x+\sin^2(3x),\ -\pi\leq x\leq\pi$

   Solution. A Fourier series is a sum of cosine and sine terms. We already have cosine and sine terms here, we just need to rewrite $\sin^2(3x)$ and we are done:
$$\begin{aligned}f(x)&=1-\sin x+3\cos 2x+\sin^2(3x)=1-\sin x+3\cos 2x+\frac{1}{2}(1-\cos 6x)\\&=\frac{3}{2}-\sin x+3\cos 2x-\frac{1}{2}\cos 6x\end{aligned}$$
$\square$

   (b) $f(x)=x^2,\ -p\leq x\leq p$

   Solution. f is even so it only has cosine terms in its Fourier series:
$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}x\right).$$
Since we are on the interval $-p\leq x\leq p$, $L=p$. Find a_0 and a_n (use integration by parts for a_n):
$$\begin{aligned}a_0&=\frac{1}{p}\int_0^p f(x)dx=\frac{1}{p}\int_0^p x^2dx=\frac{x^3}{3p}\Big|_0^p=\frac{p^2}{3}\\a_n&=\frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)dx=\frac{2}{p}\int_0^p x^2\cos\left(\frac{n\pi}{p}x\right)dx\\&=\frac{2}{p}\left[x^2\frac{p}{n\pi}\sin\left(\frac{n\pi}{p}x\right)+2x\left(\frac{p}{n\pi}\right)^2\cos\left(\frac{n\pi}{p}x\right)-2\left(\frac{p}{n\pi}\right)^3\sin\left(\frac{n\pi}{p}x\right)\right]\Big|_0^p\\&=\frac{4p^2}{(n\pi)^2}\cos n\pi=\frac{4p^2}{(n\pi)^2}(-1)^n.\end{aligned}$$
Thus
$$f(x)=\frac{p^2}{3}+\sum_{n=0}^{\infty}\frac{4p^2}{(n\pi)^2}(-1)^n\cos\left(\frac{n\pi}{p}x\right).$$
$\square$

(c) $f(x) = x\cos x$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$
Hint: This is a tough one. Use the fact that the function is odd. Also use the formula $\cos(\alpha)\sin(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$.

Solution. f is odd, so it only has sine terms in its Fourier series:

$$f(x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L}x\right).$$

Since we are on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $L = \frac{\pi}{2}$. Find b_n:

$$b_n = \frac{2}{\frac{\pi}{2}} \int_0^{\pi/2} f(x)\sin\left(\frac{n\pi}{\frac{\pi}{2}}x\right)dx = \frac{4}{\pi}\int_0^{\pi/2} x\cos x\sin(2nx)dx$$

Use the given formula on $\cos x \sin(2nx)$, then evaluate using integration by parts:

$$\begin{aligned}
b_n &= \frac{4}{\pi}\int_0^{\pi/2} x\cos x\sin(2nx)dx = \frac{4}{\pi}\int_0^{\pi/2} x\cdot\frac{1}{2}\left(\sin(2n+1)x + \sin(2n-1)x\right)dx \\
&= \frac{2}{\pi}\left[x\left(-\frac{\cos(2n+1)x}{2n+1} - \frac{\cos(2n-1)x}{2n-1}\right) + \frac{\sin(2n+1)x}{(2n+1)^2} + \frac{\sin(2n-1)x}{(2n-1)^2}\right]\Big|_0^{\pi/2} \\
&= \frac{2}{\pi}\left[\frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)^2} + \frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{(2n+1)^2} + \frac{(-1)^n}{(2n-1)^2}\right] \\
&= \frac{2}{\pi}(-1)^n\left(\frac{-1}{(2n+1)^2} + \frac{1}{(2n-1)^2}\right) = \frac{2}{\pi}(-1)^n\left(\frac{(2n+1)^2-(2n-1)^2}{(2n+1)^2(2n-1)^2}\right) \\
&= \frac{16}{\pi}(-1)^{n+1}\frac{n}{(2n+1)^2(2n-1)^2}.
\end{aligned}$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\frac{16}{\pi}(-1)^{n+1}\frac{n}{(2n+1)^2(2n-1)^2}\sin(2nx).$$

□

8. Find the cosine and sine series of the function $f(x) = x$, $0 < x < 1$.

Solution. We are only given half of our periodic function.

For the cosine series, we need to find the even expansion of $f(x) = x$ which is $f(x) = |x|$ for $-1 < x < 1$. In this case, $L = 1$. The cosine series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx = \int_0^1 xdx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}. \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos(n\pi x)dx = 2\int_0^1 x\cos(n\pi x)dx = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]\Big|_0^1 \\
&= \frac{2}{(n\pi)^2}(\cos(n\pi) - 1).
\end{aligned}$$

When n is even, $\cos(n\pi) = 1$ which implies that $a_n = 0$. When n is odd, $\cos(n\pi) = -1$, which implies that

$$a_n = \frac{-4}{(n\pi)^2}.$$

Thus the cosine series is

$$f(x) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{-4}{(n\pi)^2} \cos(n\pi x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{-4}{(2k+1)^2\pi^2} \cos(2k+1)x.$$

For the sine series, we need to find the odd expansion of $f(x)$ which is $f(x) = x$ for $-1 < x < 1$. In this case, $L = 1$. The sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin(n\pi x) = 2\int_0^1 x\sin(n\pi x)dx = 2\left[\frac{-x}{n\pi}\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\right]\Big|_0^1 \\ &= -\frac{2}{n\pi}\cos(n\pi) = -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}. \end{aligned}$$

Thus the sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x).$$

□

9. Find the Fourier transforms of the following functions:

(a) $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a. \end{cases}$

Solution.

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-a} 0e^{-i\omega x}dx + \int_{-a}^{a} 1\cdot e^{-i\omega x}dx + \int_a^{\infty} 0e^{-i\omega x}dx\right) \\ &= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{-i\omega}e^{-i\omega x}\Big|_{-a}^{a} = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{-i\omega}\left(e^{-i\omega a} - e^{i\omega x}\right) \\ &= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{-i\omega}\left(-2i\sin\omega a\right) = \sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega} \end{aligned}$$

Note: we used the equation $\dfrac{e^{ix} - e^{-ix}}{2i} = \sin x$ to simplify. □

(b) $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0. \end{cases}$

Solution.

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} 0e^{-i\omega x}dx + \int_0^{\infty} e^{-x}e^{-i\omega x}dx\right] = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-(1+i\omega)x}dx \\ &= \frac{1}{\sqrt{2\pi}}\frac{e^{-(1+i\omega)x}}{-(1+i\omega)}\Big|_0^{\infty} = \frac{1}{\sqrt{2\pi}}\left(\lim_{x\to\infty}\frac{e^{-(1+i\omega)x}}{-(1+i\omega)} - \frac{1}{\sqrt{2\pi}}\frac{1}{-(1+i\omega)}\right) \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\omega} = \frac{1}{\sqrt{2\pi}}\frac{1-i\omega}{1+\omega^2}. \end{aligned}$$

Note: we rationalized the denominator in the last line by multiplying the fraction by $1 - i\omega$. □

(c) $f(x) = \begin{cases} 1 - x^2, & |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

Solution. Use integration by parts to evaluate the integral.

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} 0e^{-i\omega x}dx + \int_{-1}^{1}(1-x^2)e^{-i\omega x}dx + \int_{1}^{\infty} 0e^{-i\omega x}dx\right] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{-i\omega x} \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{-(1-x^2)}{i\omega}e^{-i\omega x} - \frac{2x}{(i\omega)^2}e^{-i\omega x} + \frac{2}{(i\omega)^3}e^{-i\omega x}\right]\Big|_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{(i\omega)^2}(e^{i\omega} - e^{-i\omega}) + \frac{2}{(i\omega)^3}(e^{i\omega} - e^{i\omega})\right] \\
&= \frac{1}{\sqrt{2\pi}}\frac{2}{(i\omega)^3}\left[(i\omega)2i\sin\omega + 2i\sin\omega\right] = \sqrt{\frac{2}{\pi}}\frac{1}{(i\omega)^3}2i(i\omega+1)\sin\omega = -\sqrt{\frac{2}{\pi}}\frac{2(i\omega+1)\sin\omega}{\omega^3}
\end{aligned}$$

□

10. Solve the system $u_{tt} = 16u_{xx}, 0 \leq x \leq 1, t \geq 0; u(0,t) = u(1,t) = 0; u(x,0) = f(x), u_t(x,0) = 0$ where

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \leq 1. \end{cases}$$

Solution. This is a wave equation system. We are given $c = 4$, $L = 1$, $g(x) = 0$, and $f(x)$ as described above. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(b_n \cos\lambda_n t + b_n^* \sin\lambda_n t\right)$$

where

$$\begin{aligned}
b_n &= 2\int_0^1 f(x)\sin(n\pi x)dx \\
b_n^* &= \frac{2}{4n\pi}\int_0^1 g(x)\sin(n\pi x)dx \\
\lambda_n &= 4n\pi.
\end{aligned}$$

$b_n^* = 0$ since $g(x) = 0$. Find b_n (use integration by parts):

$$\begin{aligned}
b_n &= 2\left[\int_0^{1/2} 2x\sin(n\pi x)dx + \int_{1/2}^{1} 2(1-x)\sin(n\pi x)dx\right] \\
&= 2\left[\left(-\frac{2x}{n\pi}\cos(n\pi x) + \frac{2}{(n\pi)^2}\sin(n\pi x)\right)\Big|_0^{1/2} + \left(-\frac{2(1-x)}{n\pi}\cos(n\pi x) - \frac{2}{(n\pi)^2}\sin(n\pi x)\right)\Big|_{1/2}^{1}\right] \\
&= \frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin(n\pi x)\cos(4n\pi t).$$

You could clean this up more if you want to. When n is even $\sin\left(\frac{n\pi}{2}\right) = 0$, so only the odd terms contribute. When $n = 4k+1$, $\sin\left(\frac{n\pi}{2}\right) = 1$. When $n = 4k+3$, $\sin\left(\frac{n\pi}{2}\right) = -1$. □

11. Solve the system $u_{tt} = u_{xx}, u(x,0) = \sin\pi x + 3\sin 2\pi x, u_t(x,0) = \sin\pi x, -\infty < x < \infty, t \geq 0.$

Solution. We are given $c = 1$, $f(x) = \sin \pi x + 3 \sin 2\pi x$, $g(x) = \sin \pi x$. The solution to the system is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x-t) + f(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds \\ &= \frac{1}{2}\left[\sin \pi(x-t) + 3\sin 2\pi(x-t) + \sin \pi(x+t) + 3\sin 2\pi(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \sin \pi s ds \end{aligned}$$

Evaluate the right hand integral and use the identities $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$, $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ to simplify:

$$\begin{aligned} u(x,t) = \frac{1}{2}\Big[&\sin \pi x \cos \pi t - \sin \pi t \cos \pi x + 3\sin 2\pi x \cos 2\pi t - 3 \sin 2\pi t \cos 2\pi x \\ &+ \sin \pi x \cos \pi t + \sin \pi t \cos \pi x + 3 \sin 2\pi x \cos 2\pi t + 3 \sin 2\pi t \cos 2\pi x\Big] - \frac{1}{2\pi}\cos \pi s\Big|_{s=x-t}^{x+t} \\ = &\frac{1}{2}\Big[2\sin \pi x \cos \pi t + 6 \sin 2\pi x \cos 2\pi t\Big] - \frac{1}{2\pi}\left[\cos \pi(x+t) - \cos \pi(x-t)\right] \\ = &\sin \pi x \cos \pi t + 3 \sin 2\pi x \cos 2\pi t - \frac{1}{2\pi}\left[\cos \pi x \cos \pi t - \sin \pi x \sin \pi t - \cos \pi x \cos \pi t - \sin \pi x \sin \pi t\right] \\ = &\sin \pi x \cos \pi t + 3 \sin 2\pi x \cos 2\pi t + \frac{1}{\pi}\sin \pi x \sin \pi t \end{aligned}$$

□

12. Solve the system $u_t = u_{xx}, 0 \le x \le 1, t \ge 0; u(0,t) = u(1,t) = 0; u(x,0) = e^{-x}$.

Solution. This is a heat equation system. We are given $c = 1, L = 1, f(x) = e^{-x}$. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(n\pi x)$$

where

$$\begin{aligned} \lambda_n &= n\pi \\ b_n &= 2\int_0^1 f(x)\sin(n\pi x)dx = 2\int_0^1 e^{-x}\sin(n\pi x)dx. \end{aligned}$$

The integral in b_n is a repetitive integral, meaning we need to use integration by parts a couple of times to evaluate it. Choose $u = e^{-x}, dv = \sin(n\pi x)dx$:

$$\int e^{-x}\sin(n\pi x)dx = -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \int \frac{1}{n\pi}e^{-x}\cos(n\pi x)dx$$

Do integration by parts again with $u = e^{-x}, dv = \cos(n\pi x)dx$:

$$\begin{aligned} \int e^{-x}\sin(n\pi x)dx &= -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{n\pi}\left(\frac{1}{n\pi}e^{-x}\sin(n\pi x) + \int \frac{1}{n\pi}e^{-x}\sin(n\pi x)dx\right) \\ &= -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x) - \frac{1}{(n\pi)^2}e^{x}\sin(n\pi x)dx. \end{aligned}$$

Notice the integral on the right hand side is the same as the left hand side. Add it to the left hand side:

$$\left(1 + \frac{1}{(n\pi)^2}\right)\int e^{-x}\sin(n\pi x)dx = -\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x)$$

Solve for the integral:

$$\int e^{-x}\sin(n\pi x)dx = \frac{1}{1 + \frac{1}{(n\pi)^2}}\left(-\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x)\right)$$

Therefore

$$\begin{aligned} b_n &= 2\int_0^1 e^{-x}\sin(n\pi x)dx = \frac{2}{1+\frac{1}{(n\pi)^2}}\left(-\frac{1}{n\pi}e^{-x}\cos(n\pi x) - \frac{1}{(n\pi)^2}e^{-x}\sin(n\pi x)\right)\Big|_0^1 \\ &= \frac{2(n\pi)^2}{(n\pi)^2+1}\left(-\frac{1}{n\pi}\right)\left(e^{-1}\cos(n\pi)+1\right) = -\frac{2n\pi}{(n\pi)^2+1}\left(\frac{(-1)^n}{e}+1\right). \end{aligned}$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{2n\pi}{(n\pi)^2+1}\left(\frac{(-1)^n}{e}+1\right)e^{-(n\pi)^2 t}\sin(n\pi x).$$

$\square$