Math 33A – Week 5

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Name: KEY

1. Let $T(\mathbf{x}) = \begin{pmatrix} 1 & -3 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \mathbf{x}$.

   (a) Find $\text{im}(T)$ and $\ker(T)$.

   $\left(\begin{array}{cccc|c} 1 & -3 & 0 & -5 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$ already in rref.

   $\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ & \text{free} & & \text{free} \end{array}$

   1st and 3rd columns span imT

   $\boxed{\text{im}T = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}}$

   x_2 free, $x_2 = t$
   x_4 free, $x_4 = s$

   $x_1 - 3x_2 - 5x_4 = 0 \rightarrow x_1 = 3x_2 + 5x_4$
   $x_3 + 2x_4 = 0 \longrightarrow x_3 = -2x_4$

   $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3t+5s \\ t \\ -2s \\ s \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \end{pmatrix} s$$

   $\ker T = \text{span}\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$

   (b) Verify the rank-nullity theorem for this transformation.

   rank = 2
   nullity = 2

   rank + nullity = 4 = # of columns of $\begin{pmatrix} 1 & -3 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ ✓

   (c) Are there any vectors $\mathbf{y} \in \mathbb{R}^2$ such that $\mathbf{y} \notin \text{im}(T)$? *Remark: This illustrates that $\text{im}T = \mathbb{R}^2$

   One way: for $\vec{y} \in \text{im}T$ means $T(\vec{x}) = \vec{y}$ for some $\vec{x}$

   $\Rightarrow \left(\begin{array}{cccc|c} 1 & -3 & 0 & -5 & y_1 \\ 0 & 0 & 1 & 2 & y_2 \end{array}\right)$

   Q: is it possible for this system to have no solution?
   A: No, this system always has a solution no matter the value of $\vec{y}$

   $\boxed{\text{No, no } \vec{y} \text{ such that } \vec{y} \notin \text{im}T}$

   Another way: in (a) we found $\text{im}T = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

   For $y \in \text{im}T$ means there are a, b s.t. $a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \vec{y}$

   $\Rightarrow \left(\begin{array}{cc|c} 1 & 0 & y_1 \\ 0 & 1 & y_2 \end{array}\right)$

   Q: is it possible for this system to have no solution?
   A: no, this system always has a solution no matter the value of $\vec{y}$.

   $\boxed{\text{No, no } \vec{y} \text{ such that } \vec{y} \notin \text{im}T}$

2. Recall that $T(\mathbf{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}$ is the transformation that projects $\mathbf{x} = (x, y)$ onto the y-axis.

   (a) Without solving explicitly, what do you expect $\text{im}(T)$ and $\ker(T)$ to be? (You don't need to use set notation here, you can write a short description about what you think they will be.)

   $\text{im}T$ = all outputs of $T(\vec{x})$ = y-axis
   $\ker T$ = all inputs $\vec{x}$ s.t. $T(\vec{x}) = \vec{0}$ $\Rightarrow$ $\ker T$ = x-axis

   (b) Solve $\text{im}(T)$ and $\ker(T)$ explicitly. Do these match your intuition in (a)?

   $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

   $\begin{array}{cc} \uparrow & \uparrow \\ \text{free} & \end{array}$

   2nd column of original matrix spans imT

   x_1 free, $x_1 = t$
   $x_2 = 0$

   $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t$

   $\boxed{\ker T = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}}$

   ↑ this is the x-axis!

   $\boxed{\text{im}T = \text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}}$

   ↑ this is the y-axis

(c) Verify the rank-nullity theorem for this transformation.

rank $=1$ rank + nullity $=2=$ # of columns of $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

nullity $=1$

(d) Find a vector $\mathbf{y}$ such that $\mathbf{y} \notin \text{im}(T)$. * answers will vary

$\vec{y} \in \text{im}T$ means $T(\vec{x}) = \vec{y}$ for some $\vec{x}$

$\left(\begin{array}{cc|c} 0 & 0 & y_1 \\ 0 & 1 & y_2 \end{array}\right)$ If y_1 is a nonzero # then this system has no solution!

e.g. $\vec{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ since the vector $\vec{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \text{im}T$ since $\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$ has no solution

(e) Is $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ invertible? no, $\det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 \cdot 1 - 0 \cdot 0 = 0$

3. Recall that $T(\mathbf{x}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\vec{x}$ is the transformation that rotates $\mathbf{x}$ counterclockwise by $\frac{\pi}{4}$ radians and stretches the rotated vector by a factor of $\sqrt{2}$.

(a) Without solving explicitly, what do you expect $\text{im}(T)$ and $\ker(T)$ to be? (You don't need to use set notation here, you can write a short description about what you think they will be.)

$\text{im}T =$ all outputs $= \mathbb{R}^2$ since if we rotate $\mathbb{R}^2$ by $\pi/4$ and scale by $\sqrt{2}$ we still get $\mathbb{R}^2$

$\ker T =$ all inputs $\vec{x}$ s.t. $T(\vec{x}) = \vec{0} \Rightarrow \ker T = \{\vec{0}\}$

(b) Solve $\text{im}(T)$ and $\ker(T)$ explicitly. Do these match your intuition in (a)?

$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{-R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1+R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$ ← this system has the unique solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

1st and 2nd columns of original matrix span imT

$\Rightarrow \boxed{\ker T = \{\vec{0}\}}$

$\boxed{\text{im}T = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}}$ ← this is $\mathbb{R}^2$

(c) Verify the rank-nullity theorem for this transformation.

rank $=2$ rank + nullity $=2=$ # of columns of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

nullity $=0$

(d) Are there any vectors $\mathbf{y} \in \mathbb{R}^2$ such that $\mathbf{y} \notin \text{im}(T)$? * Remark: this illustrates that $\text{im}T = \mathbb{R}^2$

One way: $\vec{y} \in \text{im}T$ means $T(\vec{x}) = \vec{y}$ for some $\vec{x}$

$\left(\begin{array}{cc|c} 1 & -1 & y_1 \\ 1 & 1 & y_2 \end{array}\right) \xrightarrow{\text{row operations}} \left(\begin{array}{cc|c} 1 & 0 & \text{stuff} \\ 0 & 1 & \vdots \end{array}\right)$

this system always has a solution $\Rightarrow \boxed{\text{No } \vec{y} \text{ s.t. } \vec{y} \notin \text{im}T}$

Another way: in (b), $\text{im}T = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

$\vec{y} \in \text{im}T$ means there are a, b s.t. $a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{y}$

$\left(\begin{array}{cc|c} 1 & -1 & y_1 \\ 1 & 1 & y_2 \end{array}\right)$ ← this system always has a solution after doing row operations

$\Rightarrow \boxed{\text{No } \vec{y} \text{ s.t. } \vec{y} \notin \text{im}T}$

(e) Is $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ invertible? yes, $\det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 1 \cdot 1 - (-1) \cdot 1 = 1 + 1 = 2 \neq 0$

4. The invertibility of a square matrix is related to its rank, nullity, kernel, and image. The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in Exercise 2 is not invertible, but $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ in Exercise 3 is invertible. Fill in the following statements below (you can use Exercises 2 and 3 for reference):

   (a) If A is an $n \times n$ matrix and $\text{rank}(A) =$ __n__, then $\text{im}(A) = \mathbb{R}^n$.

   (b) If A is an $n \times n$ matrix and $\text{rank}(A) = n$, then $\text{nullity}(A) =$ __0__.

   (c) If A is an $n \times n$ matrix and $\text{nullity}(A) =$ __0__, then $\ker(A) = \{\mathbf{0}\}$.

   (d) If A is an $n \times n$ matrix and $\ker(A) =$ __$\{\vec{0}\}$__, then A is invertible.

   (e) If A is an $n \times n$ matrix and $\text{rank}(A) \neq$ __n__, A is not invertible.

   (f) If A is an $n \times n$ matrix and $\text{nullity}(A) \neq$ __0__, A is not invertible.

   (g) If A is an $n \times n$ matrix and $\ker(A) \neq \{\mathbf{0}\}$, A is __not invertible__.