Math 33B Worksheet 5 (Forced Harmonic Motion)

Name: KEY Score: ____________

Circle the name of your TA: Ziheng Nicholas Victoria

Circle the day of your discussion: Tuesday Thursday

Overview: This week we focus on "forced harmonic motion". Mathematically, the external force constructs an inhomogeneous term in the second order ODE.

We will start from last week's RLC circuit example, adding a sinusoidal force. In undamped case, when $\omega_0 \approx \omega$, **beats** or **resonance** may occur. In damped case, the solution will finally go to a **steady state**.

Then we look at different types of external forces. Typically there are two methods for solving the particular solution: **undetermined coefficients** and **variation of parameters**.

1. **RLC Circuit**

Recall the "RLC circuit" example from last week. By Kirchhoff's law we have equation

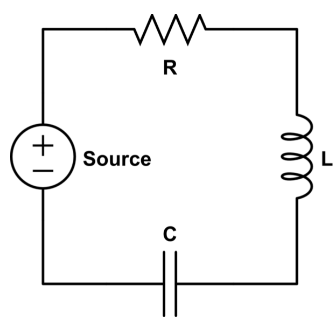
$$L\frac{dI}{dt} + IR + \frac{Q}{C} = V_{in} \tag{1}$$

This time we have sinusoidal source voltage $V_{in} = V_0 \sin \omega t$.

Since $I = \frac{dQ}{dt}$, take derivative to equation (1), we get

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \omega V_0 \cos \omega t \tag{2}$$

For simplicity, in following questions we assume initial condition $I(0) = I'(0) = 0$.



(a) Derivation of undamped solution.

First let's see pure LC circuit (R=0). Remember our natural frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, we can simplify this equation into ($F = \frac{\omega V_0}{L}$)

$$\begin{cases} \dfrac{d^2I}{dt^2} + \omega_0^2 I = F \cos \omega t \\ I(0) = I'(0) = 0 \end{cases} \tag{3}$$

Now try to solve equation (3) by trying cosine solution. How does the solution look like when $\omega < \omega_0$, $\omega = \omega_0$, $\omega > \omega_0$?

I_h: $I'' + \omega_0^2 I = 0$

$m^2 + \omega_0^2 = 0$

$m^2 = -\omega_0^2$

$m = \pm i\omega_0$

$I_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

(undetermined coeff)

If $\omega \neq \omega_0$: try $I_p = A\cos\omega t$

$I_p' = -A\omega \sin\omega t$

$I_p'' = -A\omega^2 \cos\omega t$

$-A\omega^2 \cos\omega t + \omega_0^2 A \cos\omega t = F\cos\omega t$

$A(\omega_0^2 - \omega^2) = F$

$A = \frac{F}{\omega_0^2 - \omega^2}$

$I_p = \frac{F}{\omega_0^2 - \omega^2} \cos\omega t$

$$\boxed{I = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F}{\omega_0^2 - \omega^2}\cos\omega t \quad \omega \neq \omega_0}$$

(variation of parameters)

If $\omega = \omega_0$, $W = \begin{vmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\omega_0 \sin\omega_0 t & \omega_0 \cos\omega_0 t \end{vmatrix} = \omega_0(\cos^2\omega_0 t + \sin^2\omega_0 t) = \omega_0$

$W_1 = \begin{vmatrix} 0 & \sin\omega_0 t \\ F\cos\omega_0 t & \omega_0 \cos\omega_0 t \end{vmatrix} = -F\sin\omega_0 t \cos\omega_0 t = -\frac{F}{2}\sin 2\omega_0 t$

$W_2 = \begin{vmatrix} \cos\omega_0 t & 0 \\ -\omega_0 \sin\omega_0 t & F\cos\omega_0 t \end{vmatrix} = F\cos^2\omega_0 t$

$u_1' = \frac{W_1}{W} = \frac{-F}{2\omega_0}\sin(2\omega_0 t) \longrightarrow u_1 = \int \frac{-F}{2\omega_0}\sin(2\omega_0 t)\,dt = \frac{F\cos(2\omega_0 t)}{4\omega_0^2}$

$u_2' = \frac{W_2}{W} = \frac{F\cos^2\omega_0 t}{\omega_0} \longrightarrow u_2 = \frac{F}{\omega_0}\int \frac{1}{2}(1+\cos 2\omega_0 t)\,dt = \frac{F}{2\omega_0}\left(t + \frac{\sin 2\omega_0 t}{2\omega_0}\right)$

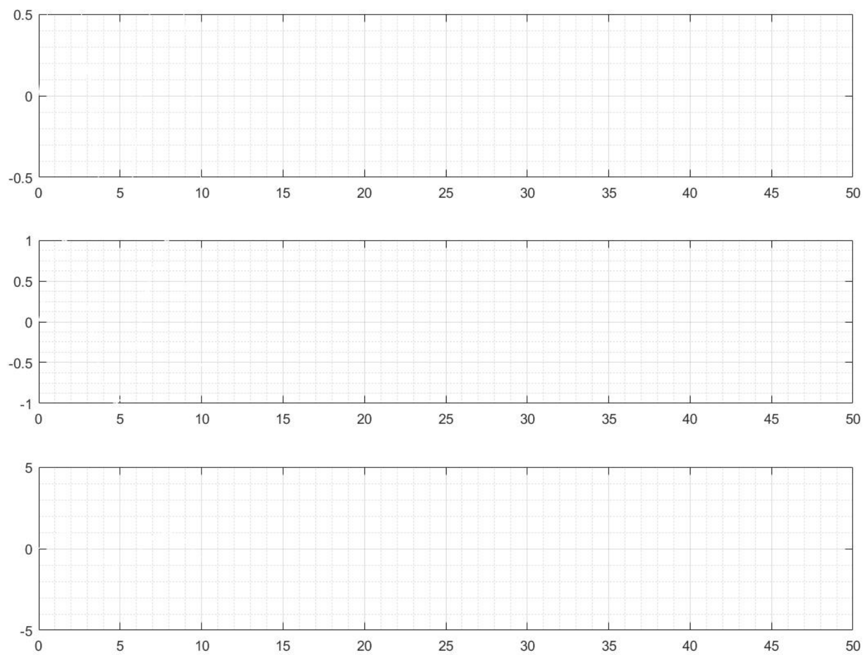
$= \frac{Ft}{2\omega_0} + \frac{F}{4\omega_0^2}\sin 2\omega_0 t$

$I_p = \cos\omega_0 t\left(\frac{F}{4\omega_0^2}\cos 2\omega_0 t\right) + \sin\omega_0 t\left(\frac{Ft}{2\omega_0} + \frac{F}{4\omega_0^2}\sin 2\omega_0 t\right)$, $\boxed{I = c_1\cos\omega_0 t + c_2 \sin\omega_0 t + I_p \quad , \omega = \omega_0}$

(b) Beats and resonance.

* see link on webpage

If $\omega_0 = \pi, F = 1$, consider different ratio $\frac{\omega}{\omega_0} = 1.2, 1.1, 1.0$. Roughly plot solutions in following graphs. Also draw envelope in dash line. How many waves are in each beats? How does amplitude change as $\omega \to \omega_0$? And how is the resonance motion different from beats?



(c) (Optional) Damped forced motion

Let's come back to RLC circuit. Try to derive the general solution to equation (2). What is the steady state?

(Hint: General solution of inhomogeneous equation is

$$y = y_p + C_1 y_1 + C_2 y_2$$

where y_p is particular solution and y_1, y_2 are solutions of homogeneous equation. You may skip this one if you feel too hard.)

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \omega V_0 \cos \omega t$$

* see Worksheet 5 for homogeneous solution

guess for $y_p = A\cos\omega t + B\sin\omega t$ (as long as $R \neq 0$)

Plug in and find coeff...

2. Now we look at different types of external forces and practice finding the practical solution

(a) Find a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin 3t \tag{4}$$

(Hint: Try $y(t) = a\cos 3t + b\sin 3t$)

y_h: $y'' + 2y' - 3y = 0$

$m^2 + 2m - 3 = 0$

$(m+3)(m-1) = 0$

$m_1 = -3,\ m_2 = 1$

$y_h = c_1 e^{-3t} + c_2 e^{t}$

y_p: $y_p = A\sin 3t + B\cos 3t$

$y_p' = 3A\cos 3t - 3B\sin 3t$

$y_p'' = -9A\sin 3t - 9B\cos 3t$

Plug in: $-9A\sin 3t - 9B\cos 3t + 2(3A\cos 3t - 3B\sin 3t) - 3(A\sin 3t + B\cos 3t) = 5\sin 3t$

$(-9A - 6B - 3A)\sin 3t + (-9B + 6A - 3B)\cos 3t = 5\sin 3t$

$(-12A - 6B)\sin 3t + (-12B + 6A)\cos 3t = 5\sin 3t + 0\cos 3t$

$$\begin{cases} -12A - 6B = 5 \\ 6A - 12B = 0 \to A = 2B \end{cases}$$

$-24B - 6B = 5$

$-30B = 5$

$B = -1/6,\ A = -1/3$

$y_p = -\frac{1}{3}\sin 3t - \frac{1}{6}\cos 3t$

$\boxed{y = c_1 e^{-3t} + c_2 e^{t} - \frac{1}{3}\sin 3t - \frac{1}{6}\cos 3t}$

(b) Find a particular solution to the equation

$$y'' - y' - 2y = 3e^{-t} \tag{5}$$

(Hint: Try $y(t) = ate^{-t}$, or use variation of parameters)

y_h: $y'' - y' - 2y = 0$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m_1 = 2,\ m_2 = -1$

$y_h = c_1 e^{2t} + c_2 e^{-t}$

Undetermined coeff: $\cancel{y_p = Ae^{-t}}$

$y_p = Ate^{-t}$

$y_p' = Ae^{-t} - Ate^{-t}$

$y_p'' = -2Ae^{-t} + Ate^{-t}$

plug in:

$-2Ae^{-t} + \cancel{Ate^{-t}} - (Ae^{-t} - \cancel{Ate^{-t}}) - 2\cancel{Ate^{-t}} = 3e^{-t}$

$-3Ae^{-t} = 3e^{-t}$

$-3A = 3$

$A = -1$

$\boxed{y_p = -te^{-t}}$

Variation of Parameters:

$W = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -e^{t} - 2e^{t} = -3e^{t}$

$W_1 = \begin{vmatrix} 0 & e^{-t} \\ 3e^{-t} & -e^{-t} \end{vmatrix} = -3e^{-2t}$

$W_2 = \begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & 3e^{-t} \end{vmatrix} = 3e^{t}$

$u_1' = \frac{W_1}{W} = \frac{-3e^{-2t}}{-3e^{t}} = e^{-t}$, $u_1 = \int e^{-t}\,dt = -e^{-t}$

$u_2' = \frac{W_2}{W} = \frac{3e^{t}}{-3e^{t}} = -1$, $u_2 = \int -1\,dt = -t$

$y_p = u_1 y_1 + u_2 y_2 = -e^{-t}\cdot e^{2t} - te^{-t} \longrightarrow \boxed{y_p = -e^{t} - te^{-t} \text{ or } y_p = -te^{-t}}$

(c) Find a particular solution to the equation

$$y'' + y = \tan t \tag{6}$$

(Hint: Try $y(t) = v_1 \cos t + v_2 \sin t$ and use variation of parameters)

y_h: $y'' + y = 0$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$y_h = c_1 \cos t + c_2 \sin t$

$y_1 = \cos t$, $y_2 = \sin t$

$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

$W_1 = \begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix} = -\sin t \tan t = -\frac{\sin^2 t}{\cos t}$

$W_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix} = \cos t \tan t = \sin t$

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$u_1' = \frac{W_1}{W} = -\frac{\sin^2 t}{\cos t}$

$u_1 = \int -\frac{\sin^2 t}{\cos t} dt = \int -\frac{(1-\cos^2 t)}{\cos t} dt = -\int (\sec t - \cos t) dt = -\ln|\sec t + \tan t| + \sin t$

$u_2' = \frac{W_2}{W} = \sin t$

$u_2 = \int \sin t \, dt = -\cos t$

$y_p = u_1 y_1 + u_2 y_2 = \cos t \left(-\ln|\sec t + \tan t| \cancel{+ \sin t}\right) + \sin t \left(\cancel{-\cos t}\right) = -\ln|\sec t + \tan t|$

$\boxed{y = c_1 \cos t + c_2 \sin t - \ln|\sec t + \tan t|}$