Math 33B Worksheet 6 (Linear System)

Name: KEY Score: ____________

Circle the name of your TA: Ziheng Nicholas Victoria

Circle the day of your discussion: Tuesday Thursday

Overview: This week we focus on "linear system". Having a knowledge on linear algebra is highly recommended for learning this section. Our basic technique for solving linear ODE system is by finding the eigenvalues and eigenvectors of a matrix.

1. **Solving linear system of ODE**

   Consider 2×2 linear system $y' = Ay$, with

   (a) $A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$ (c) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

   First try to find the eigenvalues and eigenvectors, write down $A = P\Lambda P^{-1}$. Then make substitution $y = Px$ to solve the linear system of ODE. Describe the long-term behaviour of your solutions. How does the eigenvalues influence the behaviour of solution?

What the prompt says: $\vec{y}' = A\vec{y}$ $A = P\Lambda P^{-1}$ where $P = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\vec{y}' = P\Lambda P^{-1}\vec{y}$

$P^{-1}\vec{y}' = \Lambda P^{-1}\vec{y}$

$(P^{-1}\vec{y})' = \Lambda(P^{-1}\vec{y})$ take $\vec{x} = P^{-1}\vec{y} \Rightarrow \vec{y} = P\vec{x}$

$\vec{x}' = \Lambda\vec{x}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \end{cases}$ separable equations

solution: $x_1 = e^{\lambda_1 t}, x_2 = e^{\lambda_2 t}$

$\vec{y} = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{pmatrix}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' =$ multiply out and solve for $y_1, y_2 \ldots$

Let's solve the same way as section 9.2.

(a) $A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$

① $\begin{vmatrix} -\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda) - 6 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2) = 0$

$\lambda_1 = 3$ $\lambda_2 = -2$

② $\lambda_1 = 3$: $\left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right)$ $3x_1 = 2x_2$, $x_1 = \frac{2}{3}x_2$ $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda_2 = -2$: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right)$ $2x_1 = -2x_2$, $x_1 = -x_2$ $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

③ $\boxed{\vec{x} = c_1 e^{3t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

(b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$

① $\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) + 4 = 12 - 8\lambda + \lambda^2 + 4 = \lambda^2 - 8\lambda + 16$

$= (\lambda-4)^2$ $\lambda = 4$ mult 2

② $\left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right)$ $-x_1 + 2x_2 = 0$, $x_1 = 2x_2$ $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\left(\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right)$ $-x_1 + 2x_2 = 1$, $x_1 = 2x_2 - 1$ $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

③ $\boxed{\vec{x} = c_1 e^{4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{4t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}$

(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

① $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

② $\lambda = i$: $\left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right)$ $-x_1 - ix_2 = 0$, $x_1 = -ix_2$ $\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ * note: $\vec{v}_2$ not necessary

③ $e^{\lambda_1 t}\vec{v}_1 = e^{it}\begin{pmatrix} -i \\ 1 \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i\cos t - i^2\sin t \\ \cos t + i\sin t \end{pmatrix} = \begin{pmatrix} -i\cos t + \sin t \\ \cos t + i\sin t \end{pmatrix}$

$= \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

$\boxed{\vec{x} = c_1\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}}$

2. **Mass-spring problem revisited**

Recall the equation of mass-spring system:

$$mx''(t) = -kx(t) - \mu x'(t) + F(t) \tag{1}$$

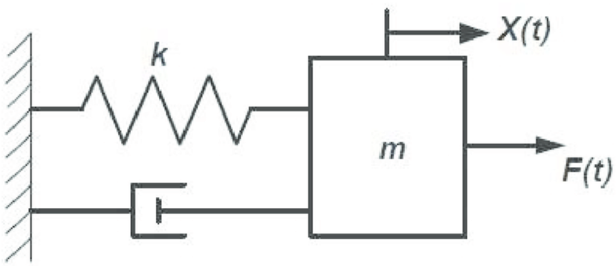

Figure 1: Mass on a spring

By subtitution $x'(t) = v(t)$, we get a 2D linear system of ODE:

$$\begin{cases} x'(t) = v(t) \\ v'(t) = -\frac{k}{m}x(t) - \frac{\mu}{m}v(t) + \frac{F(t)}{m} \end{cases} \tag{2}$$

We can rewrite it in matrix form

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{F(t)}{m} \end{pmatrix} \tag{3}$$

(a) Suppose $F(t) = 0$, can you derive the same solution as we did in unforced harmonic motion? (Recall the 3 types of solutions with different damping coefficient.) * See Worksheet 5

solution omitted. → (b) (Optional) Suppose $\mu = 0, F(t) = F\cos(\omega t)$, can you derive the same solution as we did in forced harmonic motion? (Recall beats and resonance.)

(a) $\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$

1 $\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{\mu}{m}-\lambda \end{vmatrix} = -\lambda\left(-\frac{\mu}{m}-\lambda\right) + \frac{k}{m} = \lambda^2 + \frac{\mu}{m}\lambda + \frac{k}{m} = 0$

$m\lambda^2 + \mu\lambda + k = 0$

$\lambda = \frac{-\mu \pm \sqrt{\mu^2-4km}}{2m}$

(i) e-values are real, distinct if $\mu^2-4km>0$ (over-damped)

$\lambda_1 = \frac{-\mu+\sqrt{\mu^2-4km}}{2m}$, $\lambda_2 = \frac{-\mu-\sqrt{\mu^2-4km}}{2m}$

(ii) " " real, repeated if $\mu^2-4km=0$ (critically damped)

$\lambda = -\frac{\mu}{2m}$ mult. 2

(iii) " " complex if $\mu^2-4km<0$ (underdamped)

$\lambda = \frac{-\mu \pm \sqrt{(-1)(4km-\mu^2)}}{2m} = \frac{-\mu \pm i\sqrt{4km-\mu^2}}{2m}$

$\lambda_1 = \frac{-\mu+i\sqrt{4km-\mu^2}}{m}$, $\lambda_2 = \frac{-\mu-\sqrt{4km-\mu^2}}{2m}$

2 (i) $\lambda_1 = \frac{-\mu+\sqrt{\mu^2-4km}}{2m}$ $\left(\begin{array}{cc|c} -\left(\frac{-\mu+\sqrt{\mu^2-4km}}{2m}\right) & 1 & 0 \\ -\frac{k}{m} & -\frac{\mu}{m}+\frac{\mu-\sqrt{\mu^2-4km}}{2m} & 0 \end{array}\right)$ $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-\mu+\sqrt{\mu^2-4km}}{2m} \end{pmatrix}$

$\lambda_2 = \frac{-\mu-\sqrt{\mu^2-4km}}{2m}$ $\left(\begin{array}{cc|c} -\left(\frac{-\mu-\sqrt{\mu^2-4km}}{2m}\right) & 1 & 0 \\ -\frac{k}{m} & -\frac{\mu}{m}+\frac{\mu+\sqrt{\mu^2-4km}}{2m} & 0 \end{array}\right)$ $\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{-\mu-\sqrt{\mu^2-4km}}{2m} \end{pmatrix}$

$\begin{pmatrix} x \\ v \end{pmatrix} = c_1 e^{(-\mu+\sqrt{\mu^2-4km})t/2m}\begin{pmatrix} 1 \\ \frac{-\mu+\sqrt{\mu^2-4km}}{2m} \end{pmatrix} + c_2 e^{(-\mu-\sqrt{\mu^2-4km})t/2m}\begin{pmatrix} 1 \\ \frac{-\mu-\sqrt{\mu^2-4km}}{2m} \end{pmatrix}$

* 1st row is same solution we had in Week 5

- You can repeat w/ the other two cases and check that the first row is the same as solutions in Week 5