Math 33B Worksheet 4 (Harmonic Motion)

Name: KEY Score: ____________

Circle the name of your TA: Ziheng Nicholas Victoria

Circle the day of your discussion: Tuesday Thursday

Overview: Second order linear ODE with constant coefficients usually describes a harmonic motion. When there is no damping force, the solution is a periodic cosine funciton. When there is a damping force, the solution can be **underdamped**, **critically damped**, or **overdamped**, depending on the coefficients.

Now we are going to learn harmonic motion through two examples in physics.

1. **RLC Circuit**

Recall in first week's worksheet we discussed simple "RC circuit", where the circuit is composed of resistors (R) and capacitors (C). Now we introduce inductor (L):

	R	L	C
V=	IR	$L\frac{dI}{dt}$	$\frac{Q}{C}$

Figure 1 below illustrates the "RLC circuit". By Kirchhoff's law we have equation

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = 0 \tag{1}$$

(Here source voltage $V_{in} = 0$ because we only study homogeneous equation.)

Since $I = \frac{dQ}{dt}$, take derivative to equation (1), we get

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0 \tag{2}$$

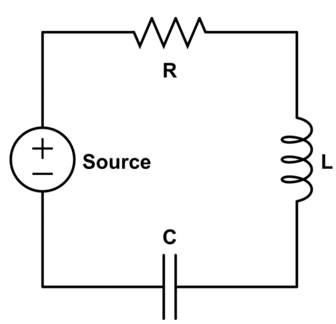


Figure 1: RLC circuit

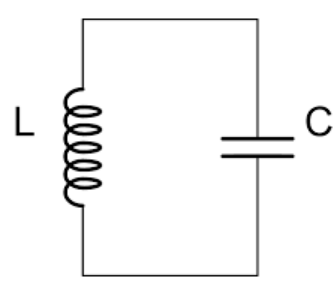


Figure 2: LC circuit

* assume $L, R, C > 0$

(a) If there is no resistor (R), it is "LC circuit" (as shown in figure 2), which satisfies

$$L\frac{d^2I}{dt^2} + \frac{I}{C} = 0 \tag{3}$$

Given initial condition $I(0) = A\cos(\phi)$, $I'(0) = -\frac{A}{\sqrt{LC}}\sin(\phi)$, try to find solution to (3). Explain the meaning of A, ϕ and $\omega_0 = \frac{1}{\sqrt{LC}}$.

(Hint: Try cosine solution.)

Characteristic equation: $L m^2 + \frac{1}{C} = 0$

$$m^2 = -\frac{1}{LC}$$

$$m = \pm\sqrt{\frac{-1}{LC}}$$

$$m = \pm i\sqrt{\frac{1}{LC}}$$

$$I(t) = c_1 \cos\left(\frac{1}{\sqrt{LC}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{LC}}t\right)$$

$I(0) = A\cos\varphi$: $A\cos\varphi = c_1\cdot 1 + c_2\cdot 0$

$$c_1 = A\cos\varphi$$

$$I(t) = A\cos\varphi\cos\left(\frac{1}{\sqrt{LC}}t\right) + c_2\sin\left(\frac{1}{\sqrt{LC}}t\right)$$

$$I'(t) = -\frac{A\cos\varphi}{\sqrt{LC}}\sin\left(\frac{1}{\sqrt{LC}}t\right) + \frac{c_2}{\sqrt{LC}}\cos\left(\frac{1}{\sqrt{LC}}t\right)$$

$I'(0) = -\frac{A}{\sqrt{LC}}\sin\varphi$: $-\frac{A}{\sqrt{LC}}\sin\varphi = 0 + \frac{c_2}{\sqrt{LC}}\cdot 1$

$$c_2 = -A\sin\varphi$$

$$I(t) = A\cos\varphi\cos\left(\frac{t}{\sqrt{LC}}\right) - A\sin\varphi\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\boxed{I(t) = A\cos\left(\frac{t}{\sqrt{LC}} + \varphi\right)}$$

since $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(b) Let's come back to RLC circuit. Try to find the general solution to equation (2). You may need to divide into cases according to parameters.

(Hint: Try complex exponential solution.)

$$LI'' + RI' + \frac{I}{C} = 0$$

Characteristic equation: $Lm^2 + Rm + \frac{1}{C} = 0 \Rightarrow m = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$

• If $R^2 - \frac{4L}{C} < 0$ then $m = \frac{-R \pm \sqrt{-1}\sqrt{\frac{4L}{C} - R^2}}{2L} = \frac{-R}{2L} \pm i\frac{\sqrt{\frac{4L}{C} - R^2}}{2L}$

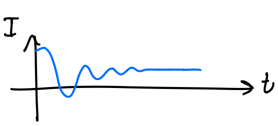
$$\Rightarrow \boxed{I(t) = e^{-Rt/2L}\left[c_1\cos\left(\frac{\sqrt{\frac{4L}{C}-R^2}}{2L}t\right) + c_2\sin\left(\frac{\sqrt{\frac{4L}{C}-R^2}}{2L}t\right)\right]}$$

• If $R^2 - \frac{4L}{C} = 0$ then $m = \frac{-R \pm 0}{2L}$ multiplicity 2 $\Rightarrow \boxed{I(t) = c_1 e^{-Rt/2L} + c_2 t e^{-Rt/2L}}$

• If $R^2 - \frac{4L}{C} > 0$ then $\boxed{I(t) = c_1 e^{\left(\frac{-R + \sqrt{R^2 - 4L/C}}{2L}\right)t} + c_2 e^{\left(\frac{-R - \sqrt{R^2 - 4L/C}}{2L}\right)t}}$

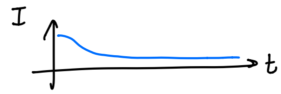
(c) In part (b), classify solutions as "underdamped", "overdamped" and "critical damped". Try to explain their long term behavior. How does the behavior of solution depend on parameters?

$R^2 - \frac{4L}{C} < 0$ underdamped



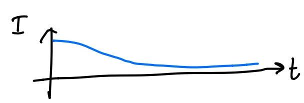
oscillations

$R^2 - \frac{4L}{C} = 0$ critically damped



No oscillations

$R^2 - \frac{4L}{C} > 0$ overdamped



No oscillations

similar behaviors

2. **Mass on a Spring**

Let's look at another example of harmonic motion. Consider a spring-mass system (shown below). The motion can be described in following equation:

$$m\frac{d^2x}{dt^2} = -kx + D(x') + F(t) \tag{4}$$

where $D(x')$ is damping force, and $F(t)$ is externel force. Again, since we are talking about homogeneous equation, here assume $F(t) = 0$.

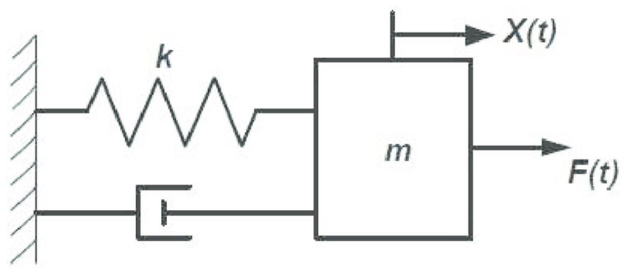


Figure 3: Mass on a spring

* assuming k,m>

(a) If there is no damping force

$$m\frac{d^2x}{dt^2} = -kx \tag{5}$$

Given initial condition $x(0) = x_0, x'(0) = v_0$, try to find the solution to (5). What are the "amplitude", "phase angle" and "natural frequency"?

$mx'' + kx = 0$

Characteristic equation: $m\lambda^2 + k = 0$

$\lambda^2 = -\frac{k}{m}$

$\lambda = \pm\sqrt{-\frac{k}{m}}$

$\lambda = \pm i\sqrt{\frac{k}{m}}$

$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$x(0) = x_0:\quad x_0 = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = x_0$

$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$x'(t) = x_0\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\,t\right) + c_2\sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}}\,t\right)$

$x'(0) = v_0 \Rightarrow \quad v_0 = c_2\sqrt{\frac{k}{m}} \quad\Rightarrow\quad c_2 = v_0\sqrt{\frac{m}{k}}$

$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + v_0\sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$$\boxed{x(t) = \underbrace{\sqrt{x_0^2 + v_0^2\left(\frac{m}{k}\right)}}_{\text{amplitude}} \cos\left(\underbrace{\sqrt{\frac{k}{m}}}_{\text{natural frequency}}\,t - \underbrace{\tan^{-1}\left(\frac{v_0\sqrt{\frac{m}{k}}}{x_0}\right)}_{\text{phase angle}}\right)}$$

(b) Let's come back to equation (4) with damping force $D(v) = -\mu v$, i.e.

$$m\frac{d^2x}{dt^2} = -kx - \mu x' \tag{6}$$

Try to find the general solution to (6). You may need to divide into cases according to parameters.

$mx'' + \mu x' + kx = 0$ characteristic equation: $m\lambda^2 + \mu\lambda + k = 0 \Rightarrow \lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$

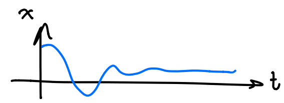
If $\mu^2 - 4mk < 0$ then $\lambda = \frac{-\mu \pm i\sqrt{4mk - \mu^2}}{2m} \Rightarrow \boxed{x(t) = e^{-\mu t/2m}\left(c_1 \cos\left(\frac{\sqrt{4mk-\mu^2}}{2m}\,t\right) + c_2 \sin\left(\frac{\sqrt{4mk-\mu^2}}{2m}\,t\right)\right)}$

If $\mu^2 - 4mk = 0$ then $\lambda = \frac{-\mu \pm 0}{2m}$ mult.2 $\Rightarrow \boxed{x(t) = c_1 e^{-\mu t/2m} + c_2 t e^{-\mu t/2m}}$

If $\mu^2 - 4mk > 0$ then $\boxed{x(t) = c_1 e^{\left(\frac{-\mu + \sqrt{\mu^2 - 4mk}}{2m}\right)t} + c_2 e^{\left(\frac{-\mu - \sqrt{\mu^2 - 4mk}}{2m}\right)t}}$

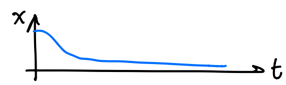
(c) In part (b), classify solutions as "underdamped", "overdamped" and "critical damped". Try to explain their long term behavior. How does the behavior of solution depend on parameters?

$\mu^2 - 4mk < 0$ Underdamped



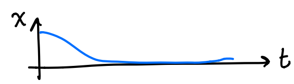
oscillations

$\mu^2 - 4mk = 0$ critically damped



no oscillations

$\mu^2 - 4mk > 0$ overdamped



no oscillations

similar behavior

3. Compare your solution in problem 1 and 2. Did you see they are actually the same math problem? What general conclusion can you get about harmonic motion?