Math 33B Worksheet 7 (Phase Plane for Linear Systems)

Name: KEY ______________________ Score: ________

Circle the name of your TA: Ziheng Nicholas Victoria

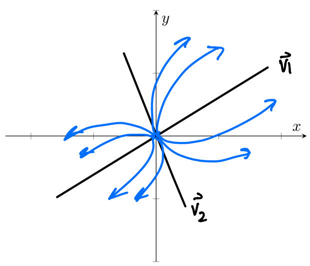
Circle the day of your discussion: Tuesday Thursday

1. The linear system

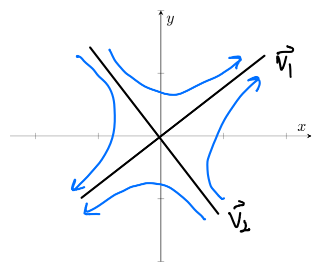
$$x' = ax + by$$
$$y' = cx + dy$$

can be written in matrix form $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can use the eigenvalues and eigenvectors of A to sketch the phase plane/portrait for the system. Below are the possible behaviors for linear systems with nonzero eigenvalues, sketch a plausible phase portrait for each behavior.

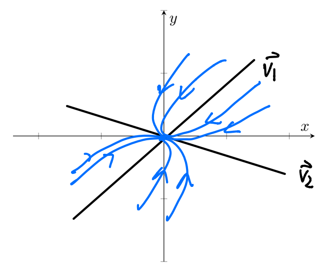
(i) Real, distinct eigenvalues ($\lambda_1 \neq \lambda_2$)



unstable node
($\lambda_1 > \lambda_2 > 0$)

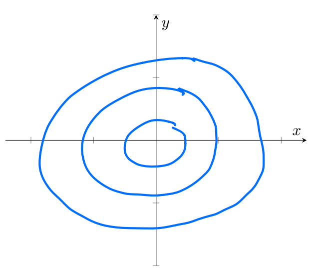


saddle
($\lambda_1 > 0 > \lambda_2$)

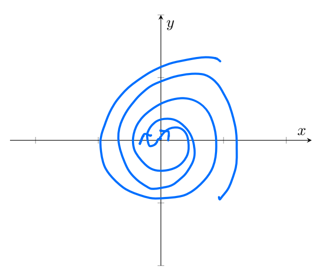


stable node
($\lambda_1 < \lambda_2 < 0$)

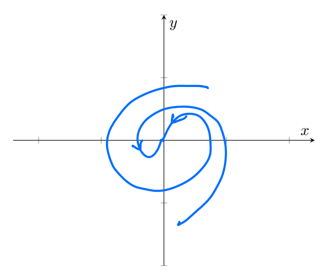
(ii) Complex eigenvalues ($\lambda = \alpha \pm \beta i$)



center
($\alpha = 0$)



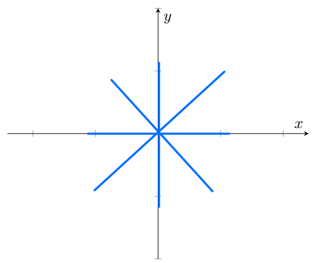
stable spiral
($\alpha < 0$)



unstable spiral
($\alpha > 0$)

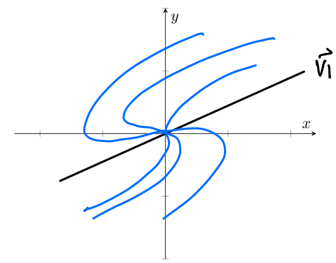
Question: How do we determine whether the center or spiral is clockwise or counter-clockwise?

(iii) Real repeated eigenvalues



star node

Occurs when $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$



degenerate node

Occurs when $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$

Question: When are each of these nodes stable or unstable?

2. Consider the linear system

$$x' = x + y$$
$$y' = 4x - 2y.$$

(a) Rewrite this system in matrix form $\mathbf{x}' = A\mathbf{x}$. Find the eigenvalues of A and use these to classify the behavior and stability of the system.

$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$ $\quad \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = -2 - \lambda + 2\lambda + \lambda^2 - 4 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$

$\lambda_1 = -3 \quad \lambda_2 = 2$

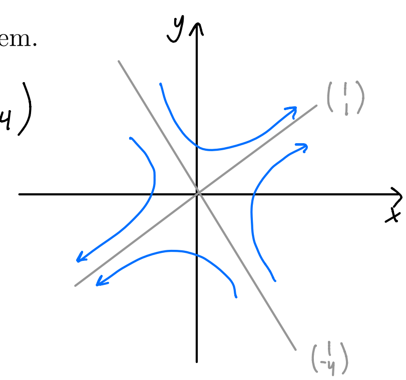
opposite signs $\Rightarrow$ $\boxed{\text{saddle unstable}}$

(b) Find the eigenvectors of A and sketch the phase portrait of the system.

$\lambda_1 = -3$: $\left(\begin{array}{cc|c} 1-(-3) & 1 & 0 \\ 4 & -2-(-3) & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right)$ $\quad 4x_1 + x_2 = 0$, $x_2 = -4x_1$ $\quad \vec{v_1} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\lambda_2 = 2$: $\left(\begin{array}{cc|c} 1-2 & 1 & 0 \\ 4 & -2-2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right)$ $\quad -x_1 + x_2 = 0$, $x_1 = x_2$ $\quad \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

dominant eigenvector



3. Consider the linear system

$$x' = 4x - y$$
$$y' = 2x + y.$$

(a) Rewrite this system in matrix form $\mathbf{x}' = A\mathbf{x}$. Find the eigenvalues of A and use these to classify the behavior and stability of the system.

$\vec{x}' = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \vec{x}$

$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)+2 = 4-4\lambda-\lambda+\lambda^2+2 = \lambda^2-5\lambda+6 = (\lambda-3)(\lambda-2)$

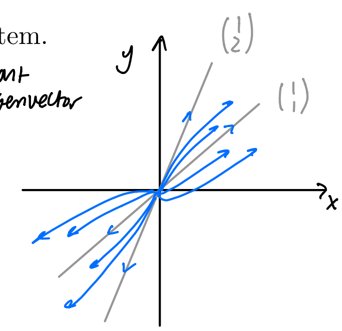
$\lambda_1 = 3 \quad \lambda_2 = 2$

same sign $\longrightarrow$ $\boxed{\text{unstable node}}$

(b) Find the eigenvectors of A and sketch the phase portrait of the system.

$\lambda_1 = 3$ $\left(\begin{array}{cc|c} 4-3 & -1 & 0 \\ 2 & 1-3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right)$ $x_1 = x_2$ $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ dominant eigenvector

$\lambda_2 = 2$: $\left(\begin{array}{cc|c} 4-2 & -1 & 0 \\ 2 & 1-2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right)$ $x_2 = 2x_1$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



4. Consider the linear system

$$x' = x - y$$
$$y' = x + y.$$

(a) Rewrite this system in matrix form $\mathbf{x}' = A\mathbf{x}$. Find the eigenvalues of A and use these to classify the behavior and stability of the system.

$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}$

$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 1 + 1 = \lambda^2 - 2\lambda + 2$

$\lambda = \dfrac{2 \pm \sqrt{4-4(1)(2)}}{2(1)} = \dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm 2i}{2}$

$\lambda = 1 \pm i$

$\operatorname{Re}\lambda = 1 > 0$ $\boxed{\text{unstable spiral}}$

(b) The eigenvalues in (a) are complex with nonzero real part. Come up with a way to determine whether the spiral is clockwise or counterclockwise, then sketch the phase portrait of the system.

Plug in points to see what vector field is doing.

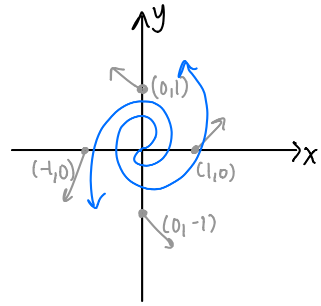
$x' = x - y$
$y' = x + y$

At $(1,0)$ $x' = 1$, $y' = 1$ i.e. right and up

At $(0,1)$ $x' = -1$, $y' = 1$ i.e. left and up

At $(-1,0)$ $x' = -1$, $y' = -1$ i.e. left and down

At $(0,-1)$, $x' = 1, y' = -1$ i.e. right and down



counterclockwise

5. **Simple Harmonic Oscillator.** The vibrations of a mass hanging from a linear spring are governed by the linear differential equation

$$mx'' + kx = 0$$

where m is the mass, k is the spring constant, and x is the displacement of the mass from the equilibrium. We can rewrite this second order differential equation as the first order system

$$x' = v$$
$$v' = -\omega^2 x$$

where $\omega^2 = k/m$. Plot the phase portrait for this system. What do these closed orbits represent?

$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$ eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{vmatrix} = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$

$Re(\lambda) = 0$ center

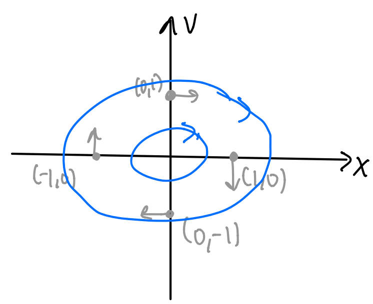
Clockwise or counterclockwise?
plug in points

At $(1,0)$ $x' = 0$, $y' = -\omega^2$ i.e. down

At $(0,1)$ $x' = 1$, $v' = 0$ i.e. right

At $(-1,0)$ $x' = 0$, $v' = \omega^2$ i.e. up

At $(0,-1)$ $x' = -1$, $v' = 0$ i.e. left



clockwise

Closed orbits represent the oscillations of the spring – spring moves up and down, velocity of spring also oscillates from + to −.