Math 33B Worksheet 9 (Review)

Name: KEY Score: ____________

Circle the name of your TA: Ziheng Nicholas Victoria

Circle the day of your discussion: Tuesday Thursday

Directions: For Exercises 1-9, match its type using A-H. No term is used twice. Then solve Exercises 1-9.

A. First order separable equation

B. First order linear equation

C. Exact equation

D. Second order homogeneous equation

E. One dimensional phase portraits and stability

F. Second order nonhomogeneous equation, undetermined coefficients

G. Second order nonhomogeneous equation, variation of parameters

H. Two dimensional homogeneous linear systems and phase portraits

I. Higher order homogeneous linear systems

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1. G Solve $y'' - 2y' + y = e^t \arctan t$.
2. B Find the general solution of $(1+x)y' - xy = x + x^2$.
3. D Solve the following differential equations.

   (a) $2y'' - 5y' - 3y = 0$ (b) $y'' - 10y' + 25y = 0$ (c) $y'' + 4y' + 7y = 0$

4. A Find an explicit solution of the given initial value problem: $x^2 \frac{dy}{dx} = y - xy,\ y(-1) = -1$.
5. H For each of the following systems, (i) Draw a phase portrait for the system, (ii) classify the stability of (0,0), (iii) Find the general solution.

   (a) $\mathbf{X}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \mathbf{X}$ (b) $\mathbf{X}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{X}$ (c) $\dfrac{dx}{dt} = 2x + 3y$, $\dfrac{dy}{dt} = 2x + y$

6. F Solve $y'' - 2y' - 3y = 4x - 5 + 6e^{2x}$.
7. C Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2$.
8. E Consider the differential equation $\frac{dP}{dt} = P(a - bP)$ where a, b are positive constants.

   (a) Find the equilibrium points.

   (b) Draw a one-dimensional phase portrait, use this to classify the stability of the equilibrium points found in (a).

   (c) What is the expected behavior of $P(t)$ as $t \to \infty$?

9. I Solve $\mathbf{X}' = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X},\ \mathbf{X}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.

① Homogeneous: $y''-2y'+y=0$

$m^2-2m+1=0$

$(m-1)^2=0$

$m=1$ mult 2 $\longrightarrow$ $y=c_1e^t+c_2te^t$

$y_1=e^t \quad y_2=te^t$

Variation of Parameters: $W=\begin{vmatrix} e^t & te^t \\ e^t & e^t+te^t \end{vmatrix} = e^t(e^t+te^t)-e^t\cdot te^t = e^{2t}$

$W_1=\begin{vmatrix} 0 & te^t \\ e^t\arctan t & e^t+te^t \end{vmatrix} = 0- te^{2t}\arctan t = -te^{2t}\arctan t$

$W_2=\begin{vmatrix} e^t & 0 \\ e^t & e^t\arctan t \end{vmatrix} = e^{2t}\arctan t - 0 = e^{2t}\arctan t$

$u_1' = \frac{W_1}{W} = \frac{-te^{2t}\arctan t}{e^{2t}} = -t\arctan t$

$\Rightarrow u_1 = \int -t\arctan t\,dt = -\frac{t^2}{2}\arctan t + \int \frac{t^2}{2}\cdot\frac{1}{1+t^2}dt = -\frac{t^2}{2}\arctan t + \frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt$

parts: $u=\arctan t \quad dv=-t\,dt$

$du=\frac{1}{1+t^2}dt \quad v=-\frac{t^2}{2}$

$= -\frac{t^2}{2}\arctan t + \frac{1}{2}\int\left(1-\frac{1}{1+t^2}\right)dt = -\frac{t^2}{2}\arctan t + \frac{1}{2}\left(t-\arctan t\right)$

$u_2' = \frac{W_2}{W} = \frac{e^{2t}\arctan t}{e^{2t}} = \arctan t$

$\Rightarrow u_2 = \int \arctan t\,dt = t\arctan t - \int\frac{t}{1+t^2}dt = t\arctan t - \frac{1}{2}\ln(1+t^2)$

parts: $u=\arctan t \quad dv=dt$

$du=\frac{1}{1+t^2}dt \quad v=t$

$y_p = u_1y_1+u_2y_2 = \left(-\frac{t^2}{2}\arctan t+\frac{t}{2}-\frac{1}{2}\arctan t\right)e^t + \left(t\arctan t - \frac{1}{2}\ln(1+t^2)\right)te^t$

$= \frac{t^2}{2}e^t\arctan t + \frac{t}{2}e^t - \frac{e^t}{2}\arctan t - \frac{te^t}{2}\ln(1+t^2)$

$y=y_h+y_p = \boxed{c_1e^t+c_2te^t+\frac{t^2e^t}{2}\arctan t+\frac{t}{2}e^t-\frac{e^t}{2}\arctan t-\frac{te^t}{2}\ln(1+t^2)}$

② $(1+x)y'-xy=x+x^2$

Standard form: $y'-\frac{x}{1+x}y=\frac{x(1+x)}{(1+x)} \rightarrow y'-\frac{x}{1+x}y=x \quad (*)$

* Note: can also use long division

Integrating factor: $\mu=\exp\left(\int\frac{-x}{1+x}dx\right)=\exp\left(-\int\frac{x}{1+x}dx\right)=\exp\left(-\int\frac{x+1-1}{1+x}dx\right)=\exp\left(-\int\left(\frac{x+1}{1+x}-\frac{1}{1+x}\right)dx\right)$

$=\exp\left(-\int\left(1-\frac{1}{1+x}\right)dx\right)=\exp\left(-(x-\ln|1+x|)\right)=\exp\left(\ln|1+x|-x\right)=e^{\ln|1+x|}e^{-x}=(1+x)e^{-x}$

Multiply $(*)$ by μ: $(1+x)e^{-x}\left[y'-\frac{x}{1+x}y\right]=x(1+x)e^{-x}$

$\underbrace{(1+x)e^{-x}y'-xe^{-x}y}_{\text{want to write as product rule}}=x(1+x)e^{-x} \quad (**)$

② continued

Need to check if $\left((1+x)e^{-x}\right)' \overset{?}{=} -xe^{-x}$

$$\left((1+x)e^{-x}\right)' = e^{-x} - (1+x)e^{-x} = -xe^{-x} \checkmark$$

So $(**)$ becomes

$$\left[(1+x)e^{-x}y\right]' = x(1+x)e^{-x}$$
$$(1+x)e^{-x}y = \int (x^2+x)e^{-x}\,dx$$

parts $u = x^2+x \quad dv = e^{-x}dx$
$du = (2x+1)dx \quad v = -e^{-x}$

$$= -(x^2+x)e^{-x} + \int (2x+1)e^{-x}dx$$

parts again: $u = 2x+1 \quad dv = e^{-x}dx$
$du = 2 \quad v = -e^{-x}$

$$= -(x^2+x)e^{-x} - (2x+1)e^{-x} + 2\int e^{-x}dx$$
$$= -(x^2+x)e^{-x} - (2x+1)e^{-x} - 2e^{-x} + C$$
$$= -e^{-x}\left[x^2+x+2x+1+2\right] + C$$
$$(1+x)e^{-x}y = -e^{-x}(x^2+3x+3) + C$$

$$\boxed{y = \frac{-(x^2+3x+3)}{(1+x)} + \frac{Ce^x}{(1+x)}}$$

Check: $y = \dfrac{Ce^x - x^2 - 3x - 3}{x+1}$

$$y' = \frac{(x+1)(Ce^x - 2x - 3) - (Ce^x - x^2 - 3x - 3)}{(x+1)^2} = \frac{Cxe^x - 2x^2 - \cancel{3x} + \cancel{Ce^x} - 2x - \cancel{3} - \cancel{Ce^x} + x^2 + \cancel{3x} + \cancel{3}}{(x+1)^2}$$

$$= \frac{Cxe^x - x^2 - 2x}{(x+1)^2}$$

$$(1+x)y' - xy = \frac{\cancel{Cxe^x} - x^2 - 2x - \cancel{Cxe^x} + x^3 + 3x^2 + 3x}{x+1} = \frac{x^3+2x^2+x}{x+1} = \frac{x(x^2+2x+1)}{x+1} = \frac{x(x+1)^2}{x+1}$$

$$= x(x+1) = x^2 + x \checkmark$$

③ (a) $2y'' - 5y' - 3y = 0$

characteristic equation $2m^2 - 5m - 3 = 0$

$$2m^2 - 6m + 1m - 3 = 0$$
$$2m(m-3) + 1(m-3) = 0$$
$$(2m+1)(m-3) = 0$$
$$m = -1/2,\ m = 3$$

$$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$$

(b) $y'' - 10y' + 25y = 0$

characteristic equation: $m^2 - 10m + 25 = 0$

$$(m-5)^2 = 0$$
$m = 5$ mult 2

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

(c) $y'' + 4y' + 7y = 0$

characteristic equation: $m^2 + 4m + 7 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2(1)} = \frac{-4 \pm \sqrt{-12}}{2} = \frac{-4 \pm 2i\sqrt{3}}{2}$$

$$= -2 \pm i\sqrt{3}$$

$$\boxed{y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)}$$

④ $x^2 \frac{dy}{dx} = y - xy \quad , y(-1) = -1$

$x^2 \frac{dy}{dx} = y(1-x) \longrightarrow \frac{1}{y} dy = \frac{1-x}{x^2} dx \longrightarrow \int \frac{1}{y} dy = \int \left(\frac{1}{x^2} - \frac{1}{x}\right) dx$

$\ln y = -\frac{1}{x} - \ln|x| + C$

$y = e^{-1/x} e^{-\ln|x|} e^{C} \longrightarrow y = C \frac{e^{-1/x}}{x}$

when $x=-1, y=-1$: $-1 = \frac{Ce^{1}}{-1} \Rightarrow C = \frac{1}{e}$ $\searrow$ $\boxed{y = \frac{e^{-1/x - 1}}{x}}$

Check: $y' = \frac{x(e^{-1/x-1})' - e^{-1/x-1}(x)'}{x^2} = \frac{x \cdot e^{-1/x-1} \cdot \frac{1}{x^2} - e^{-1/x-1}}{x^2} = \frac{\frac{1}{x} e^{-1/x-1} - e^{-1/x-1}}{x^2}$

$x^2 y' = \frac{1}{x} e^{-1/x-1} - e^{-1/x}$

$y - xy = \frac{e^{-1/x-1}}{x} - e^{-1/x-1}$

same! ✓

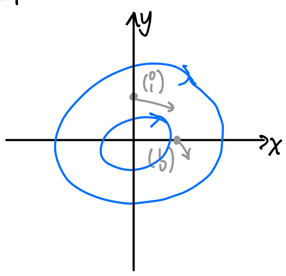
⑤ (a) $\vec{x}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \vec{x}$ eigenvalues: $\begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (-1)(8) = -4 - 2\lambda + 2\lambda + \lambda^2 + 8 = \lambda^2 + 4 = 0$

$\lambda = \pm 2i \longrightarrow$ center, stable (ii)

(i) Is center clockwise or counterclockwise?

$\begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$ right down

$\begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ right down



(iii) $\lambda = 2i$, $\left(\begin{array}{cc|c} 2-2i & 8 & 0 \\ -1 & -2-2i & 0 \end{array}\right) \rightarrow x_1 = (-2-2i)x_2$ $\quad \vec{v}_1 = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix}$

Need to find $Re(e^{\lambda t} \vec{v}_1)$ and $Im(e^{\lambda_1 t} \vec{v}_1)$:

$e^{2it}\begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = (\cos 2t + i \sin 2t)\begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2\cos 2t + 2i\cos 2t + 2i \sin 2t - 2\sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix} = \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + i \begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}$

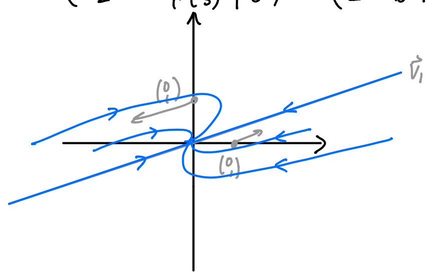
$\boxed{\vec{x} = c_1 \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}}$

(b) $\vec{x}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \vec{x}$ eigenvalues $\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (3-\lambda)(-9-\lambda) - (-18)(2) = -27 - 3\lambda + 9\lambda + \lambda^2 + 36 = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2 = 0$

$\lambda = -3$ mult 2 $\quad$ stable (ii)

(i) star or degenerate node?

$\left(\begin{array}{cc|c} 3-(-3) & -18 & 0 \\ 2 & -9-(-3) & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $x_1 = 3x_2$ $\quad \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ only one eigenvector so degenerate node



$\begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -18 \\ -9 \end{pmatrix}$ left down $\quad \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ right up

⑤ (b) continued

(iii) we found $\lambda=-3$ mult 2, $\vec{v}_1=\begin{pmatrix}3\\1\end{pmatrix}$. Find $\vec{v}_2$ using $(A-\lambda I)\vec{v}_2=\vec{v}_1$: $\left(\begin{array}{cc|c}6&-18&3\\2&-6&1\end{array}\right)\to\left(\begin{array}{cc|c}1&-3&\frac{1}{2}\\0&0&0\end{array}\right)$ $x_1=3x_2+\frac{1}{2}$, $\vec{v}_2=\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}$

General solution is $\boxed{\vec{x}=c_1e^{-3t}\begin{pmatrix}3\\1\end{pmatrix}+c_2e^{-3t}\left(\begin{pmatrix}3\\1\end{pmatrix}t+\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}\right)}$

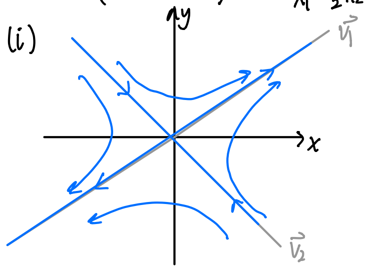
(c) $\vec{x}'=\begin{pmatrix}2&3\\2&1\end{pmatrix}\vec{x}$ eigenvalues: $\begin{vmatrix}2-\lambda&3\\2&1-\lambda\end{vmatrix}=(2-\lambda)(1-\lambda)-2(3)=2-2\lambda-\lambda+\lambda^2-6=\lambda^2-3\lambda-4=(\lambda-4)(\lambda+1)=0$

$\lambda=4,\ \lambda=-1$ saddle, unstable (ii)

$\lambda=4$: $\left(\begin{array}{cc|c}-2&3&0\\2&-3&0\end{array}\right)\to 2x_1=3x_2$, $x_1=\frac{3}{2}x_2$ $\vec{v}_1=\begin{pmatrix}3\\2\end{pmatrix}$
*dominant

$\lambda=-1$: $\left(\begin{array}{cc|c}3&3&0\\2&2&0\end{array}\right)$ $x_1=-x_2$ $\vec{v}_2=\begin{pmatrix}-1\\1\end{pmatrix}$

(i)



(iii) General solution is $\boxed{\vec{x}=c_1e^{4t}\begin{pmatrix}3\\2\end{pmatrix}+c_2e^{-t}\begin{pmatrix}-1\\1\end{pmatrix}}$

⑥ $y''-2y'-3y=4x-5+6e^{2x}$

Homogeneous: $y''-2y'-3y=0$ $\quad m^2-2m-3=0$ $\quad y_h=c_1e^{3x}+c_2e^{-x}$

$(m-3)(m+1)=0$

$m=3,-1$

Guess y_p: $y_p=Ax+B+Ce^{2x}$

$y_p'=A+2Ce^{2x}$

$y_p''=4Ce^{2x}$

Plug in: $4Ce^{2x}-2(A+2Ce^{2x})-3(Ax+B+Ce^{2x})=4x-5+6e^{2x}$

$\cancel{4Ce^{2x}}-2A-\cancel{4Ce^{2x}}-3Ax-3B-3Ce^{2x}=4x-5+6e^{2x}$

$x(-3A)+(-2A-3B)+e^{2x}(-3C)=4x-5+6e^{2x}$

so $\begin{cases}-3A=4 \to A=-4/3\\ -2A-3B=-5 \to B=-(-5+2A)/3=(5+8/3)/3=23/9\\ -3C=6\to C=-2\end{cases}$

$y_p=-\frac{4}{3}x+\frac{23}{9}-2e^{2x}$

Check: $y_p'=-\frac{4}{3}-4e^{2x}$ $\quad y_p''-2y_p'-3y_p=-8e^{2x}-2\left(-\frac{4}{3}-4e^{2x}\right)-3\left(-\frac{4}{3}x+\frac{23}{9}-2e^{2x}\right)=\frac{8}{3}+4x-\frac{23}{3}+6e^{2x}$

$y_p''=-8e^{2x}$ $\quad =4x-5+6e^{2x}$ ✓

So $\boxed{y=c_1e^{3x}+c_2e^{-2x}-\frac{4}{3}x+\frac{23}{9}-2e^{2x}}$

⑦ $\frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y(1-x^2)}$, $y(0)=2$

$(xy^2-\cos x\sin x)dx+y(x^2-1)dy=0$ $\quad M=xy^2-\cos x\sin x$ $\quad N=y(x^2-1)$

$M_y=2xy \xleftrightarrow[\text{exact!}]{} N_x=2xy$

Want $\frac{\partial f}{\partial x}=M \Rightarrow f=\int M\,dx+g(y)=\int(xy^2-\sin x\cos x)dx+g(y)=\int(xy^2-\frac{1}{2}\sin 2x)dx+g(y)=\frac{1}{2}x^2y^2+\frac{\cos 2x}{4}+g(y)$

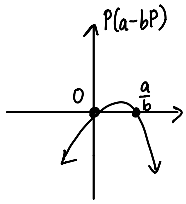
Want $\frac{\partial f}{\partial y}=N \Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{2}x^2y^2+\frac{\cos 2x}{4}+g(y)\right)=y(x^2-1)\Rightarrow \cancel{x^2y}+g'(y)=\cancel{x^2y}-y \Rightarrow g'(y)=-y \Rightarrow g(y)=-\frac{1}{2}y^2$

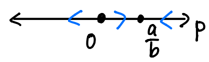
So $f=\frac{1}{2}x^2y^2+\frac{\cos 2x}{4}-\frac{1}{2}y^2 \Rightarrow \boxed{\frac{1}{2}x^2y^2+\frac{\cos 2x}{4}-\frac{1}{2}y^2=C}$

⑧ $\frac{dP}{dt} = P(a-bP)$

(a) $P(a-bP)=0 \rightarrow$ $P=0$ or $a-bP=0$, $P=\frac{a}{b}$

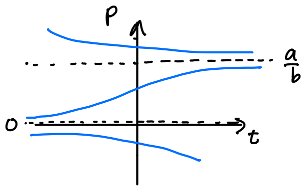
(b)





0 is unstable
$\frac{a}{b}$ is stable

(c)



as $t \to \infty$, $P(t) \to \frac{a}{b}$ if $P(0)>0$
$P(t) \to -\infty$ if $P(0)<0$

⑨ $\vec{x}' = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

eigenvalues: $\begin{vmatrix} 1-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = +(2-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)\left[(1-\lambda)^2-4\right] = (2-\lambda)\left(1-2\lambda+\lambda^2-4\right) = (2-\lambda)(\lambda^2-2\lambda-3)$

$= (2-\lambda)(\lambda-3)(\lambda+1) = 0 \rightarrow \lambda_1=2,\ \lambda_2=3,\ \lambda_3=-1$

$\lambda_1=2$: $\left(\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -5/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = +5/2\, x_3$, $x_2 = -3/2\, x_3$ $\vec{v}_1 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$

$\lambda_2=3$: $\left(\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = +2x_3$, $x_2 = 0$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3=-1$: $\left(\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = -2x_3$, $x_2 = 0$ $\vec{v}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$\vec{x} = c_1 e^{2t}\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

at $t=0$: $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \rightarrow$ 2nd row says $-3c_1 = 3 \Rightarrow c_1 = -1 \rightarrow \begin{cases} 1 = 5c_1 - 2 - 2c_3 \\ 0 = 2c_1 - 1 + c_3 \end{cases} \rightarrow \begin{cases} 3 = 5c_1 - 2c_3 \\ 0 = 2c_1 + c_3 \end{cases}$

$c_3 = -2c_1 \rightarrow \quad 3 = 5c_1 - 2(-2c_1)$

$3 = 9c_1$

$c_1 = 1/3,\ c_3 = -2/3$

Thus $\boxed{\vec{x} = \frac{1}{3}e^{2t}\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} - e^{3t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}e^{-t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}$

Check: $\vec{x}' = \frac{2}{3}e^{2t}\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} - 3e^{3t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{3}e^{-t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\vec{x} = \frac{1}{3}e^{2t}\begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} - e^{3t}\begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{3}e^{-t}\begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \frac{2}{3}e^{2t}\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} - 3e^{3t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ ✓ same!