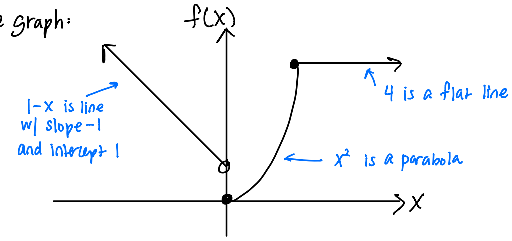
Math 31AL – PPI Solutions

① $f(x) = \begin{cases} 1-x, & x<0 \\ x^2, & 0 \le x \le 2 \\ 4, & x \geq 2 \end{cases}$

(a) Sketch the graph:



(b) * You can evaluate limits using the graph OR the given function

i. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1-x = 1$

ii. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$

iii. $\lim_{x\to 0} f(x) =$ DNE

iv. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

v. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1$

vi. $\lim_{x\to 1} f(x) = 1$

vii. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$

viii. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 4 = 4$

ix. $\lim_{x\to 2} f(x) = 4$

(c) Not continuous at $x=0$, continuous at $x=1$ and $x=2$

(d) No, because not continuous at $x=0$.

② $f(x) = \begin{cases} kx^2+2x+1, & 0 \le x \le 4 \\ 2-kx & x>4 \end{cases}$ For continuous, need $f(4) = \lim_{x\to 4} f(x) = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} kx^2+2x+1 = k(4)^2+2(4)+1 = 16k+8+1 = 16k+9$ ← this is also $f(4)$

set equal

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} 2-kx = 2-k(4) = 2-4k$

$2-4k = 16k+9$

$-7 = 20k$

$k = \frac{-7}{20}$

③ (a) $\lim_{z\to 8} z^{2/3} = \underbrace{8^{2/3} = (\sqrt[3]{8})^2 = 2^2}_{\text{all these answers ok}} = 4$

(b) $\lim_{z\to -3} x^{-2}+2x = \boxed{(-3)^{-2}+2(-3)} = \underbrace{\frac{1}{(-3)^2} - 6 = \frac{1}{9} - 6 = \frac{1}{9} - \frac{54}{9}}_{\text{all these answers ok}} = \frac{-53}{9}$

(c) $\lim_{u\to 4} \frac{\sqrt{u}-1}{\sqrt{u+5}+1} = \frac{\sqrt{4}-1}{\sqrt{4+5}+1} = \frac{2-1}{\sqrt{9}+1} = \frac{1}{3+1} = \frac{1}{4}$

(d) $\lim_{x\to -1} \frac{x^2+4x+3}{2x^2+x-1} = \frac{(-1)^2+4(-1)+3}{2(-1)^2+(-1)-1} = \frac{1-4+3}{2-1-1} = \frac{0}{0}$ ☹

Factor: x^2+4x+3

$(x+3)(x+1)$

$2x^2+x-1$

$2x^2+2x-x-1$

$2x(x+1)-1(x+1)$

$(2x-1)(x+1)$

So $\lim_{x\to -1} \frac{x^2+4x+3}{2x^2+x-1} = \lim_{x\to -1} \frac{(x+3)\cancel{(x+1)}}{(2x-1)\cancel{(x+1)}} = \lim_{x\to -1} \frac{x+3}{2x-1} = \frac{-1+3}{2(-1)-1} = \frac{2}{-2-1} = \frac{-2}{3}$

(e) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{4}-2}{4-4} = \frac{0}{0}$ ☹

Two ways! One way conjugate: $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

Another way factor: $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\cancel{\sqrt{x}-2}}{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

① continued

(f) $\lim_{x\to 1}\left(\frac{6}{x^2-1}-\frac{3}{x-1}\right)=\frac{1}{0}-\frac{3}{0}=\infty-\infty$

Get common denominator

$$\lim_{x\to 1}\left(\frac{6}{x^2-1}-\frac{3}{x-1}\right)=\lim_{x\to 1}\left(\frac{6}{(x-1)(x+1)}-\frac{3}{x-1}\cdot\frac{(x+1)}{(x+1)}\right)=\lim_{x\to 1}\frac{6-3(x+1)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{6-3x-3}{(x-1)(x+1)}=\lim_{x\to 1}\frac{-3x+3}{(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{-3\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}=\lim_{x\to 1}\frac{-3}{x+1}=\frac{-3}{2}$$

(g) $$\lim_{t\to 0}\frac{(1-\cos(4t))\sin(3t)}{t^2}=\lim_{t\to 0}\frac{1-\cos 4t}{t}\cdot\frac{\sin 3t}{t}=\lim_{t\to 0}\frac{1-\cos 4t}{t}\cdot\lim_{t\to 0}\frac{\sin 3t}{t}$$

$u=4t$, $t=u/4$ $v=3t$, $t=v/3$

$$=\lim_{u\to 0}\frac{1-\cos u}{u/4}\cdot\lim_{v\to 0}\frac{\sin v}{v/3}=\frac{1}{\frac{1}{4}}\lim_{u\to 0}\underbrace{\frac{1-\cos u}{u}}_{0}\cdot\frac{1}{\frac{1}{3}}\lim_{v\to 0}\underbrace{\frac{\sin v}{v}}_{1}=\frac{1}{\frac{1}{4}}\cdot 0\cdot\frac{1}{\frac{1}{3}}\cdot 1=0$$

(h) $$\lim_{x\to\frac{\pi}{2}}\frac{2x}{\sin x}=\frac{2\cdot\frac{\pi}{2}}{\sin\frac{\pi}{2}}=\frac{\pi}{1}=\pi$$

(i) $$\lim_{x\to 0}\frac{\csc 2x}{\csc 5x}=\lim_{x\to 0}\frac{\frac{1}{\sin 2x}}{\frac{1}{\sin 5x}}=\lim_{x\to 0}\frac{1}{\sin 2x}\cdot\frac{\sin 5x}{1}\cdot\frac{5x}{5x}\cdot\frac{2x}{2x}=\lim_{x\to 0}\frac{2x}{\sin 2x}\cdot\frac{\sin 5x}{5x}\cdot\frac{5\cancel{x}}{2\cancel{x}}=\lim_{x\to 0}\underbrace{\frac{2x}{\sin 2x}}_{1}\lim_{x\to 0}\underbrace{\frac{\sin 5x}{5x}}_{1}\cdot\frac{5}{2}$$

$$=1\cdot 1\cdot\frac{5}{2}=\frac{5}{2}$$

(j) $$\lim_{x\to\infty}\frac{3x^2-5x+10}{21-6x^2+x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{5}{x}+\frac{10}{x^2}}{\frac{21}{x^2}-6+\frac{1}{x}}=\frac{3-0+0}{0-6+0}=-\frac{3}{6}=\frac{-1}{2}$$

(k) $$\lim_{x\to\infty}\frac{e^{2x}+1}{e^{2x}-1}\cdot\frac{\frac{1}{e^{2x}}}{\frac{1}{e^{2x}}}=\lim_{x\to\infty}\frac{1+\frac{1}{e^{2x}}}{1-\frac{1}{e^{2x}}}=\frac{1+0}{1-0}=1$$

e^{2x} gets really big as $x\to\infty$

(l) $$\lim_{x\to-\infty}\frac{8x+7}{5+x^2}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{8+\frac{7}{x}}{\frac{5}{x}+x}=\frac{8+0}{0+\infty}=\frac{8}{\infty}=0$$

* you could also multiply by $\frac{1/x^2}{1/x^2}$

④ $$\lim_{x\to\infty}\frac{\sqrt{4x^2+x+1}}{3x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\sqrt{4x^2+x+1}\left(\frac{1}{\sqrt{x^2}}\right)}{3-\frac{1}{x}}=\lim_{x\to\infty}\frac{\sqrt{4+\frac{1}{x}+\frac{1}{x^2}}}{3-\frac{1}{x}}=\frac{\sqrt{4+0+0}}{3-0}=\frac{\sqrt{4}}{3}=\frac{2}{3}$$

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2+x+1}}{3x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to-\infty}\frac{\sqrt{4x^2+x+1}\left(\frac{-1}{\sqrt{x^2}}\right)}{3-\frac{1}{x}}=\lim_{x\to-\infty}\frac{-\sqrt{4+\frac{1}{x}+\frac{1}{x^2}}}{3-\frac{1}{x}}=\frac{-\sqrt{4+0+0}}{3-0}=\frac{-2}{3}$$

$\boxed{\text{The horizontal asymptotes are } y=\frac{2}{3},\ y=-2/3}$

⑤ One way: $-1\le\sin(\text{stuff})\le 1$, so
$$-1\le\sin^2\left(\tfrac{1}{x}\right)\le 1$$
$$-(4^x-1)\le(4^x-1)\sin^2\left(\tfrac{1}{x}\right)\le(4^x-1)\cdot 1$$
$$-\lim_{x\to 0}(4^x-1)\le\lim_{x\to 0}(4^x-1)\sin^2\left(\tfrac{1}{x}\right)\le\lim_{x\to 0}4^x-1$$
$$-(4^0-1)\le\ ''\ \le 4^0-1$$
$$0\le\ ''\ \le 0$$

Another way: $0\le\sin^2(\text{stuff})\le 1$, so
$$0\le\sin^2\left(\tfrac{1}{x}\right)\le 1$$
$$(4^x-1)\cdot 0\le(4^x-1)\sin^2\left(\tfrac{1}{x}\right)\le(4^x-1)\cdot 1$$
$$\underbrace{\lim_{x\to 0}0}_{=\,0}\le\lim_{x\to 0}(4^x-1)\sin^2\left(\tfrac{1}{x}\right)\le\underbrace{\lim_{x\to 0}(4^x-1)}_{=\,0}$$

either way: $\lim_{x\to 0}(4^x-1)\sin^2\left(\frac{1}{x}\right)=0$

⑥ Let $f(x)=x^2-10\sin x+4$. $f(x)$ is continuous because polynomials and $\sin x$ are continuous

$f(0)=0^2-10\sin 0+4=4>0$ $f\left(\frac{\pi}{4}\right)=\left(\frac{\pi}{4}\right)^2-10\sin\left(\frac{\pi}{4}\right)+4\approx-2.45<0$

By IVT there is a number c between 0 and $\pi/4$ s.t. $f(c)=0$.

⑦ $f(x) = x^2 - x + 2$

(a) $f(-1) = (-1)^2 - (-1) + 2 = 1 + 1 + 2 = 4$

(b) $f(-1+h) = (-1+h)^2 - (-1+h) + 2 = (-1+h)(-1+h) - (-1+h) + 2 = 1 - h - h + h^2 + 1 - h + 2 = 4 - 3h + h^2$

(c) $\dfrac{f(-1+h) - f(-1)}{h} = \dfrac{\cancel{4} - 3h + h^2 - \cancel{4}}{h} = \dfrac{-3h + h^2}{h} = \dfrac{\cancel{h}(-3+h)}{\cancel{h}} = -3+h$

(d) $\lim\limits_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h} = \lim\limits_{h \to 0} (-3+h) = -3 + 0 = -3$

(e) $f'(-1)$

⑧ $f(x) = x^{-1/2} = \dfrac{1}{\sqrt{x}}$ $\qquad f(x+h) = \dfrac{1}{\sqrt{x+h}}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{\sqrt{x}}{\sqrt{x}} \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \frac{\sqrt{x+h}}{\sqrt{x+h}}}{h} = \lim_{h \to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{\frac{h}{1}} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \cdot \frac{1}{h} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}\; h(\sqrt{x} + \sqrt{x+h})} = \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - \cancel{h}}{\sqrt{x}\sqrt{x+h}\; \cancel{h}(\sqrt{x} + \sqrt{x+h})} = \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x}\sqrt{x+0}\,(\sqrt{x} + \sqrt{x+0})}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x} \cdot 2\sqrt{x}} = \frac{-1}{2(\sqrt{x})^3} = \frac{-1}{2x^{3/2}} = \frac{-1}{2} x^{-3/2} \checkmark$$

⑨ $f(x) = x^2 + 5x - \sin x + 1$ $\qquad f(0) = 0^2 + 5(0) - \sin 0 + 1 = 0 + 0 - 0 + 1 = 1 \longrightarrow (0,1)$ is the point the line is going through

$f'(x) = 2x + 5 - \cos x + 0$ $\qquad f'(0) = 2(0) + 5 - \cos 0 = 5 - 1 = 4 \longrightarrow m = 4$

$y - y_1 = m(x - x_1) \longrightarrow y - 1 = 4(x - 0)$

$y - 1 = 4x$

$y = 4x + 1$

⑩ (a) $f(x) = \cos x - x^{100}$

$f'(x) = -\sin x - 100x^{99}$

(b) $g(x) = x \sin x$ $\qquad$ product rule

$g'(x) = (x)' \sin x + x(\sin x)' = 1 \cdot \sin x + x \cos x = \sin x + x\cos x$

(c) $h(x) = \dfrac{x^2}{\cos x}$

$$h'(x) = \frac{\cos x (x^2)' - x^2(\cos x)'}{(\cos x)^2} = \frac{\cos x \cdot 2x - x^2(-\sin x)}{\cos^2 x} = \frac{2x\cos x + x^2 \sin x}{\cos^2 x}$$

⑪ (a) F

(b) T

(c) F

(d) T

(e) F

(f) T

(g) T

(h) F