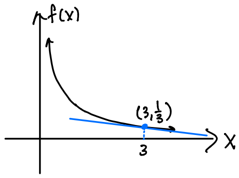
Week 4 Solutions

Math 31AL
Written by Victoria Kala, vtkala@math.ucla.edu

Tuesday, October 22

① $f'(3)$ for $f(x)=\frac{1}{x}$ $\qquad f(3)=\frac{1}{3}$, $\quad f(3+h)=\frac{1}{3+h}$

$$f'(3)=\lim_{h\to0}\frac{\frac{3}{3}\cdot\frac{1}{3+h}-\frac{1}{3}\cdot\frac{3+h}{3+h}}{h}=\lim_{h\to0}\frac{\frac{3-(3+h)}{3(3+h)}}{h}=\lim_{h\to0}\frac{\cancel{3}-\cancel{3}-\cancel{h}}{3(3+h)}\cdot\frac{1}{\cancel{h}}=\lim_{h\to0}\frac{-1}{3(3+h)}=\frac{-1}{3(3+0)}=-\frac{1}{9}$$



Tangent line: $y-f(a)=f'(a)(x-a)$

$y-f(3)=f'(3)(x-3)$

$y-\frac{1}{3}=-\frac{1}{9}(x-3)$

$y=-\frac{1}{9}x+\frac{1}{3}+\frac{1}{3}\longrightarrow y=-\frac{1}{9}x+\frac{2}{3}$

② $f(x)=x^3-12x$

$$f(x+h)=(x+h)^3-12x=\underbrace{(x+h)(x+h)}(x+h)-12(x+h)=(x^2+2xh+h^2)(x+h)-12x-12h=x^3+3x^2h+3xh^2+h^3-12x-12h$$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{12x}-12h-(\cancel{x^3}-\cancel{12x})}{h}=\lim_{h\to0}\frac{\cancel{h}(3x^2+3xh+h^2-12)}{\cancel{h}}=\lim_{h\to0}3x^2+3xh+h^2-12$$

$$=3x^2+3x(0)+0^2-12=3x^2-12$$

③ Translation: need to find when $f'(x)=0$.

From ②, $f'(x)=3x^2-12=0\longrightarrow 3(x^2-4)=0\longrightarrow 3(x-2)(x+2)=0\longrightarrow x=2,-2$

Find the points: $f(2)=2^3-12(2)=8-24=-16\longrightarrow \boxed{(2,-16)}$

$f(-2)=(-2)^3-12(-2)=-8+24=16\longrightarrow \boxed{(-2,16)}$

④ $f(x)=\frac{1}{x^2}$, $\quad f(x+h)=\frac{1}{(x+h)^2}$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\frac{x^2}{x^2}\cdot\frac{1}{(x+h)^2}-\frac{1}{x^2}\cdot\frac{(x+h)^2}{(x+h)^2}}{h}=\lim_{h\to0}\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{\frac{h}{1}}=\lim_{h\to0}\frac{x^2-(x+h)^2}{x^2(x+h)^2}\cdot\frac{1}{h}=\lim_{h\to0}\frac{\cancel{x^2}-(\cancel{x^2}+2xh+h^2)}{x^2h(x+h)^2}$$

$$=\lim_{h\to0}\frac{-\cancel{h}(2x+h)}{x^2\cancel{h}(x+h)^2}=\frac{-(2x+0)}{x^2(x+0)^2}=\frac{-2x}{x^4}=\frac{-2}{x^3}\ \checkmark$$

Thursday, Oct 24

① $$(\tan x)'=\left(\frac{\sin x}{\cos x}\right)'=\frac{\cos x(\sin x)'-\sin x(\cos x)'}{\cos^2x}=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2x}=\frac{\cos^2x+\sin^2x}{\cos^2x}=\frac{1}{\cos^2x}=\sec^2x\ \checkmark$$

② $f(x)=x^2\cos x$

$f'(x)=(x^2)'\cos x+x^2(\cos x)'=\boxed{2x\cos x-x^2\sin x}$

③ $f(x) = \sin x \cos x$

$f'(x) = (\sin x)' \cos x + \sin x (\cos x)' = \cos x \cos x - \sin x \sin x = \boxed{\cos^2 x - \sin^2 x}$

④ $y = \dfrac{\cos x}{1+\sin x}$

$y' = \dfrac{(1+\sin x)(\cos x)' - \cos x(1+\sin x)'}{(1+\sin x)^2} = \dfrac{-(1+\sin x)\sin x - \cos x \cos x}{(1+\sin x)^2} = \dfrac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \dfrac{-\sin x - 1(\cancel{\sin^2 x + \cos^2 x}^{\,1})}{(1+\sin x)^2}$

$= -\dfrac{1(\cancel{\sin x+1})}{(1+\sin x)^{\cancel{2}}} = \dfrac{-1}{1+\sin x}$

⑤ $f(x) = \tan x \csc x = \dfrac{\cancel{\sin x}}{\cos x} \cdot \dfrac{1}{\cancel{\sin x}} = \dfrac{1}{\cos x}$

$f'(x) = \dfrac{\cos x (1)' - 1(\cos x)'}{\cos^2 x} = \dfrac{\cos x \cdot 0 + \sin x}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos x} = \tan x \sec x$

⑥ $f(x) = \cos x, \quad a = \frac{\pi}{2}$

$f'(x) = -\sin x$

$f'(\frac{\pi}{2}) = -\sin \frac{\pi}{2} = -1$

$f(\frac{\pi}{2}) = \cos \frac{\pi}{2} = 0$

$y - f(a) = f'(a)(x-a)$

$y - 0 = -1(x - \frac{\pi}{2})$

$y = -x + \frac{\pi}{2}$

⑦ $y = \dfrac{\sin x + 1}{\sin x - 1}$

$y' = \dfrac{(\sin x - 1)(\sin x + 1)' - (\sin x + 1)(\sin x - 1)'}{(\sin x - 1)^2} = \dfrac{(\sin x - 1)\cos x - (\sin x + 1)\cos x}{(\sin x - 1)^2} = \dfrac{\cancel{\sin x \cos x} - \cos x - \cancel{\sin x \cos x} - \cos x}{(\sin x - 1)^2}$

$= \dfrac{-2\cos x}{(\sin x - 1)^2}$

⑧ distance = distance before brakes + distance after brakes

distance before brakes: $D = RT = 65 \dfrac{mi}{\cancel{hr}} \cdot \dfrac{1\,\cancel{hr}}{60\,\cancel{min}} \cdot \dfrac{1\,\cancel{min}}{60\,\cancel{sec}} \cdot 0.6\,\cancel{sec} = \dfrac{0.6(65)}{60^2}\,\cancel{mi} \cdot \dfrac{5280\text{ ft}}{1\,\cancel{mi}} = \dfrac{0.6(65)(5280)\text{ ft}}{60^2} = 57.2\text{ ft}$

distance after brakes: Need to find when car stopped, i.e. when $s'(t) = 0$.

$s(t) = -13t^2 + 95.3t$ (units: ft)

$s'(t) = -26t + 95.3 = 0 \longrightarrow 26t = 95.3 \rightarrow t = \dfrac{95.3}{26} = 3.6654$ sec

Distance traveled in 3.6654 sec: $s(3.6654) = -13(3.6654)^2 + 95.3(3.6654) = 174.6556$ ft

total difference: $174.6556\text{ ft} + 57.2\text{ ft} = 231.8556\text{ ft} < 235\text{ ft} \longrightarrow$ yes!