

Math 33B: First Order Equations

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Name: KEY

1. Solve the following initial value problem:

$$\frac{dy}{dx} = 2e^y x, \quad y(0) = 0$$

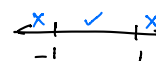
What is the interval of validity for your solution?

$$\textcircled{1} \quad e^{-y} \frac{dy}{dx} = 2x$$

$$\text{So } \boxed{y = -\ln(1-x^2)}$$

$$\textcircled{2} \quad \int e^{-y} \frac{dy}{dx} dx = \int 2x dx$$

$$\text{Domain: } \begin{aligned} 1-x^2 &> 0 \\ (1-x)(1+x) &> 0 \end{aligned}$$



$$-e^{-y} = x^2 + C$$

$$\text{Interval of validity is } \boxed{(-1, 1)}$$

$$\begin{aligned} \textcircled{3} \quad e^{-y} &= C - x^2 \rightarrow \text{at } x=0, y=0 \\ -y &= \ln(C - x^2) \\ y &= \ln(C - x^2) \end{aligned}$$

$e^0 = C - 0$
 $C = 1$

2. Solve the following first order equation:

$$y' = -\frac{3y}{x} + \frac{\sqrt{1+x^2}}{x^2}$$

$$\textcircled{1} \quad y' + \frac{3}{x} y = \frac{\sqrt{1+x^2}}{x^2}$$

$$\textcircled{4} \quad x^3 y' = \int x \sqrt{1+x^2} dx = \frac{1}{2} \int \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$$

$u = 1+x^2$
 $du = 2x dx \rightarrow \frac{du}{2} = x dx$

$$\textcircled{2} \quad P(x) = \frac{3}{x}$$

$$\int P(x) dx = \int \frac{3}{x} dx = 3 \ln|x|$$

$$\mu = e^{\int P(x) dx} = e^{3 \ln|x|} = e^{\ln|x|^3} = |x|^3$$

$$\text{Drop } | \cdot | \rightarrow \mu = x^3$$

$$x^3 y = \frac{1}{3} (1+x^2)^{3/2} + C$$

$$\textcircled{5} \quad \boxed{y = \frac{\frac{1}{3} (1+x^2)^{3/2} + C}{x^3}}$$

$$\textcircled{3} \quad x^3 \left(y' + \frac{3}{x} y \right) = \frac{\sqrt{1+x^2}}{x^2} x^3$$

$$x^3 y' + 3x^2 y = x \sqrt{1+x^2}$$

$$x^3 y' + (x^3)' y = x \sqrt{1+x^2}$$

$$(x^3 y)' = x \sqrt{1+x^2}$$

3. Solve the following exact equation:

$$y^2 e^{xy} + (e^{xy} + xye^{xy} + 1) \frac{dy}{dx} = 0$$

$$\begin{aligned} \textcircled{1} \quad M &= y^2 e^{xy} & N &= e^{xy} + xye^{xy} + 1 \\ M_y &= 2ye^{xy} + y^2 e^{xy} x & N_x &= e^{xy} \cdot y + ye^{xy} + xye^{xy} \cdot y \\ &= 2ye^{xy} + y^2 x e^{xy} & &= xy^2 e^{xy} + 2ye^{xy} \quad \checkmark \quad \text{exact!} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \text{Want } \frac{\partial f}{\partial x} = M &\rightarrow f = \int y^2 e^{xy} dx + g(y) \\ &= y^2 \frac{e^{xy}}{y} + g(y) \\ &= ye^{xy} + g(y) \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad \text{Want } \frac{\partial f}{\partial y} = N &\rightarrow \frac{\partial}{\partial y} (ye^{xy} + g(y)) = e^{xy} + xye^{xy} + 1 \\ \cancel{e^{xy}} + \cancel{ye^{xy}} \cdot x + g'(y) &= \cancel{e^{xy}} + \cancel{xy} e^{xy} + 1 \\ g'(y) &= 1 \end{aligned}$$

$$\textcircled{4} \quad g'(y) = 1 \rightarrow g(y) = \int 1 dy = y$$

$$\begin{aligned} \textcircled{5} \quad f(x,y) &= ye^{xy} + g(y) \rightarrow \boxed{ye^{xy} + y = C} \\ &= ye^{xy} + y \end{aligned}$$