Math 33B: First Order Equations

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Name: KEY

1. Solve the following initial value problem:

$$\frac{dy}{dx} = 2e^y x, \quad y(0) = 0$$

What is the interval of validity for your solution?

① $e^{-y}\dfrac{dy}{dx} = 2x$

② $\int e^{-y}\dfrac{dy}{dx}\,dx = \int 2x\,dx$

$-e^{-y} = x^2 + C$

③ $e^{-y} = C - x^2 \longrightarrow$ at $x=0, y=0$: $e^0 = C - 0$, $C = 1$

$-y = \ln(C - x^2)$

$y = \ln(C - x^2)$

So $\boxed{y = -\ln(1-x^2)}$

Domain: $1 - x^2 > 0$

$(1-x)(1+x) > 0$

$\times \quad -1 \quad \checkmark \quad 1 \quad \times$

Interval of validity is $\boxed{(-1,1)}$

2. Solve the following first order equation:

$$y' = -\frac{3y}{x} + \frac{\sqrt{1+x^2}}{x^2}$$

① $y' + \frac{3}{x}y = \dfrac{\sqrt{1+x^2}}{x^2}$

② $P(x) = \frac{3}{x}$

$\int P(x)\,dx = \int \frac{3}{x}\,dx = 3\ln|x|$

$\mu = e^{\int P(x)dx} = e^{3\ln|x|} = e^{\ln|x|^3} = |x|^3$

Drop $|\cdot| \rightarrow \mu = x^3$

③ $x^3\left(y' + \frac{3}{x}y\right) = \dfrac{\sqrt{1+x^2}}{x^2}x^3$

$x^3y' + 3x^2y = x\sqrt{1+x^2}$

$x^3y' + (x^3)'y = x\sqrt{1+x^2}$

$(x^3y)' = x\sqrt{1+x^2}$

④ $x^3y = \int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int\sqrt{u}\,du = \frac{1}{2}\frac{2}{3}u^{3/2} + C$

$u = 1+x^2$

$du = 2x\,dx \rightarrow \frac{du}{2} = x\,dx$

$x^3y = \frac{1}{3}(1+x^2)^{3/2} + C$

⑤ $\boxed{y = \dfrac{\frac{1}{3}(1+x^2)^{3/2} + C}{x^3}}$

3. Solve the following exact equation:

$$y^2 e^{xy} + (e^{xy} + xye^{xy} + 1)\frac{dy}{dx} = 0$$

① $M = y^2 e^{xy}$

$M_y = 2ye^{xy} + y^2 e^{xy} x$

$= 2ye^{xy} + y^2 x e^{xy}$

$N = e^{xy} + xye^{xy} + 1$

$N_x = e^{xy} \cdot y + ye^{xy} + xye^{xy} \cdot y$

$= xy^2 e^{xy} + 2ye^{xy}$ ✓ exact!

② Want $\frac{\partial f}{\partial x} = M \rightarrow f = \int y^2 e^{xy}\, dx + g(y)$

$= y^2 \frac{e^{xy}}{y} + g(y)$

$= ye^{xy} + g(y)$

③ Want $\frac{\partial f}{\partial y} = N \rightarrow \frac{\partial}{\partial y}\left(ye^{xy} + g(y)\right) = e^{xy} + xye^{xy} + 1$

$\cancel{e^{xy}} + \cancel{ye^{xy}} \cdot x + g'(y) = \cancel{e^{xy}} + \cancel{xy\, e^{xy}} + 1$

$g'(y) = 1$

④ $g'(y) = 1 \longrightarrow g(y) = \int 1\, dy = y$

⑤ $f(x,y) = ye^{xy} + g(y)$

$= ye^{xy} + y$ $\longrightarrow$ $\boxed{ye^{xy} + y = C}$