Practice Problems II Solutions

Math 31AL

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1 (a) $f(x) = \cos^2(x^3+1) = \left(\cos(x^3+1)\right)^2$

$f'(x) = 2\left(\cos(x^3+1)\right)^1 \cdot -\sin(x^3+1)\cdot(3x^2+1) = \boxed{-2(3x^2+1)\cos(x^3+1)\sin(x^3+1)}$

(b) $g(x) = \sqrt{x+\sin x} = (x+\sin x)^{1/2}$

$g'(x) = \frac{1}{2}(x+\sin x)^{-1/2}\cdot(1+\cos x) = \boxed{\dfrac{1+\cos x}{2\sqrt{x+\sin x}}}$

2 $2x^{1/2} + 4y^{-1/2} = xy$ Take derivative of both sides

$2\cdot\frac{1}{2}x^{-1/2} + 4\left(-\frac{1}{2}\right)y^{-3/2}\cdot y' = \underbrace{y + xy'}_{\text{product rule}}$

$x^{-1/2} - 2y^{-3/2}y' = y + xy'$ isolate y' terms

$x^{-1/2} - y = 2y^{-3/2}y' + xy'$

$x^{-1/2} - y = (2y^{-3/2}+x)y' \longrightarrow y' = \dfrac{x^{-1/2}-y}{2y^{-3/2}+x} = \dfrac{\frac{1}{\sqrt{x}} - y}{\frac{2}{(\sqrt{y})^3}+x} \overset{\text{at }(1,4)}{=} \dfrac{\frac{1}{\sqrt{1}}-4}{\frac{2}{(\sqrt{4})^3}+1} = \dfrac{1-4}{\frac{2}{8}+1}$

$= \dfrac{-3}{\frac{5}{4}} = \dfrac{-12}{5} \longleftarrow$ slope

$y - y_1 = m(x-x_1) \longrightarrow y - 4 = -\frac{12}{5}(x-1)$

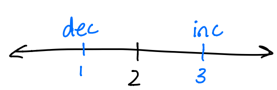
$y = -\frac{12}{5}x + \frac{12}{5} + 4$ $\boxed{y = -\frac{12}{5}x + \frac{32}{5}}$

3 $f(x) = 2x^2 - 8x + 7$

(a) $f'(x) = 4x - 8 = 0$

$4x = 8$

$x = 2$



$f'(1) = 4(1) - 8 = -4 < 0$ dec

$f'(3) = 4(3) - 8 = 12 - 8 = 4 > 0$ inc

$x=2$ is a local min

$f(2) = 2(2)^2 - 8(2) + 7$

$= 8 - 16 + 7 = -1$

$(2,-1)$ is a local min

(b) $f(2) = -1$ $\boxed{\text{min at } (2,-1)}$

$f(0) = 2(0)^2 - 8(0) + 7 = 7$

$f(5) = 2(5)^2 - 8(5) + 7 = 50 - 40 + 7 = 17$ $\boxed{\text{max at } (5,17)}$

(c) 2 is not in $[-4,1]$

$f(-4) = 2(-4)^2 - 8(-4) + 7 = 32 + 32 + 7 = 71$

$f(1) = 2(1)^2 - 8(1) + 7 = 2 - 8 + 7 = 1$

$\boxed{\begin{array}{ll}(-4,71) & \text{is max} \\ (1,1) & \text{is min}\end{array}}$

④ $f(x) = \dfrac{x^2}{x+1}$

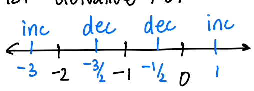
$$f'(x) = \frac{(x+1)(x^2)' - x^2(x+1)'}{(x+1)^2} = \frac{(x+1)\cdot 2x - x^2}{(x+1)^2} = \frac{2x^2+2x-x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

set num. and den. $=0$:

$x(x+2)=0$ → $x=0$; $x+2=0$, $x=-2$

$(x+1)^2=0$, $x+1=0$, $x=-1$

1st derivative test



$f'(1) = \dfrac{1(1+2)}{(1+1)^2} = \dfrac{++}{+} > 0$ inc

$f'(-\tfrac{1}{2}) = \dfrac{-\frac{1}{2}(-\frac{1}{2}+2)}{(-\frac{1}{2}+1)^2} = \dfrac{-+}{+} < 0$ dec

$f'(-\tfrac{3}{2}) = \dfrac{-\frac{3}{2}(-\frac{3}{2}+2)}{(-\frac{3}{2}+1)^2} = \dfrac{-+}{+} < 0$ dec

$f'(-3) = \dfrac{-3(-3+2)}{(-3+1)^2} = \dfrac{--}{+} > 0$ inc

local max at $x=-2$ $\quad f(-2) = \dfrac{(-2)^2}{-2+1} = \dfrac{4}{-1} = -4$

local min at $x=0$ $\quad f(0) = \dfrac{0^2}{0+1} = 0$

$\boxed{\text{local max at } (-2,-4) \quad \text{local min at } (0,0)}$

If you want to use the 2nd derivative test...

$f'(x) = \dfrac{x^2+2x}{(x+1)^2}$

$$f''(x) = \frac{(x+1)^2(x^2+2x)' - (x^2+2x)\cdot((x+1)^2)'}{(x+1)^4} = \frac{(x+1)^2(2x+2) - (x^2+2x)\cdot 2(x+1)}{(x+1)^4}$$

$$= \frac{(x+1)[(x+1)(2x+2) - 2(x^2+2x)]}{(x+1)^4} = \frac{\cancel{2x^2}+\cancel{2x}+\cancel{2x}+2-\cancel{2x^2}-\cancel{4x}}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3}$$

$f''(0) = \dfrac{2}{(0+1)^3} > 0$ concave up $\cup$ min at $x=0$

$f''(-1) = \dfrac{2}{(-1+1)^3}$ undefined

$f''(-2) = \dfrac{2}{(-2+1)^3} = -2 < 0$ concave down $\cap$ max at $x=-2$

$\boxed{\text{local min at } (0,0) \quad \text{local max at } (-2,-4)}$

⑤ $f(x) = x^6 - 9x^4$

(a) x-int: $0 = x^6 - 9x^4$

$0 = x^4(x^2-9)$

$0 = x^4(x-3)(x+3)$

$x^4=0$, $x=0$ $\quad x-3=0$, $x=3$ $\quad x+3=0$, $x=-3$

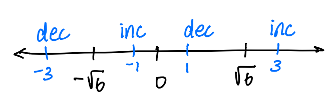
$\boxed{\text{x-int: } (0,0), (3,0), (-3,0)}$

(b) y-int: $f(0) = 0^6 - 9\cdot 0^4 = 0$ $\quad \boxed{\text{y-int: } (0,0)}$

(b) $f(x) = x^6 - 9x^4$

$f'(x) = 6x^5 - 36x^3 = 6x^3(x^2-6) = 0$

$6x^3 = 0$, $x=0$ $\quad x^2-6=0$, $x^2=6$, $x=\pm\sqrt{6}$



$\boxed{\text{Critical pts are } x=0, \pm\sqrt{6}}$

$f'(-3) = 6(-3)^3((-3)^2-6) = 6(-27)(9-6) = +-+ < 0$ dec

$f'(-1) = 6(-1)^3((-1)^2-6) = 6(-1)(-5) = +-- > 0$ inc

$f'(1) = 6(1)^3(1^2-6) = ++- < 0$ dec

$f'(3) = 6(3)^3(3^2-6) = +++ > 0$ inc

$\boxed{\begin{array}{l}\text{increasing: } (-\sqrt{6},0)\cup(\sqrt{6},\infty) \\ \text{decreasing: } (-\infty,-\sqrt{6})\cup(0,\sqrt{6})\end{array}}$

* although the problem didn't ask for extrema, we can say there is a min at $x=-\sqrt{6}$, max at $x=0$, min at $x=\sqrt{6}$

$f(\sqrt{6}) = (\sqrt{6})^6 - 9(\sqrt{6})^4 = 6^3 - 9(6)^2 = 216 - 9(36) = -108$

$f(0) = 0^6 - 9(0)^4 = 0$

$f(-\sqrt{6}) = (-\sqrt{6})^6 - 9(-\sqrt{6})^4 = 6^3 - 9(6)^2 = -108$

⑤ continued

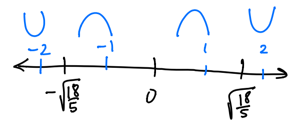
(c) $f'(x) = 6x^5 - 36x^3$

$f''(x) = 30x^4 - 108x^2 = x^2(30x^2 - 108) = 0$

$x^2 = 0$ $\quad$ $30x^2 - 108 = 0$

$x = 0$ $\quad$ $x^2 = \frac{108}{30} = \frac{18}{5}$

$x = \pm\sqrt{\frac{18}{5}} \approx \pm 1.9$



$f''(-2) = (-2)^2(30(-2)^2 - 108) = 4(120 - 108) > 0$ concave up U

$f''(-1) = (-1)^2(30(-1)^2 - 108) = 1(30 - 108) < 0$ concave down ∩

$f''(1) = 1^2(30 \cdot 1 - 108) < 0$ concave down ∩

$f''(2) = 2^2(30 \cdot 2^2 - 108) > 0$ concave up U

inflection pts at $x = \pm\sqrt{\frac{18}{5}}$

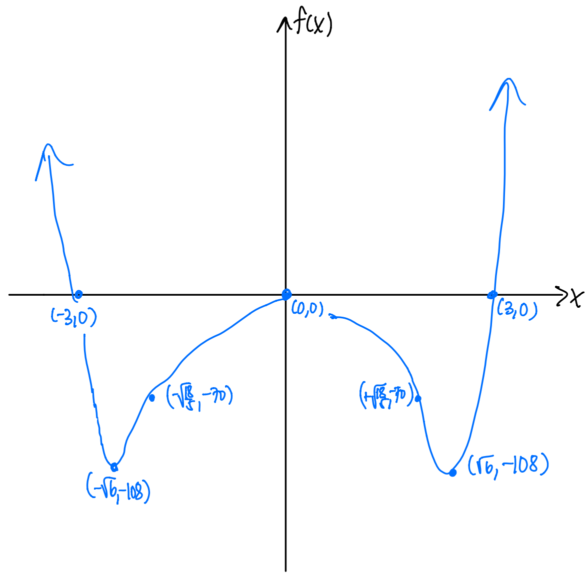
$f\left(\sqrt{\frac{18}{5}}\right) = \left(\sqrt{\frac{18}{5}}\right)^6 - 9\left(\sqrt{\frac{18}{5}}\right)^4 \approx -70$

$f\left(-\sqrt{\frac{18}{5}}\right) \approx -70$

$\boxed{\text{Concave up: } \left(-\infty, -\sqrt{\tfrac{18}{5}}\right) \cup \left(\sqrt{\tfrac{18}{5}}, \infty\right) \quad \text{Concave down: } \left(-\sqrt{\tfrac{18}{5}}, 0\right) \cup \left(0, \sqrt{\tfrac{18}{5}}\right)}$

$\boxed{\text{inflection pts} \approx (1.9, -70),\ (-1.9, -70)}$

(d)



⑥ $f(x) = \frac{x-2}{x-3}$

(a) x-int: $0 = \frac{x-2}{x-3}$ $\quad$ $\boxed{\text{x-int: } (2,0)}$ $\quad$ y-int: $f(0) = \frac{0-2}{0-3} = \frac{2}{3}$ $\quad$ $\boxed{\text{y-int } (0, \tfrac{2}{3})}$

$0 = x - 2$

$x = 2$

(b) den. = 0: $x - 3 = 0$

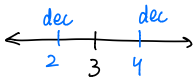
$\boxed{x = 3 \text{ is V.A.}}$

(c) $\lim_{x\to\infty} \frac{x-2}{x-3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1 - \frac{2}{x}}{1 - \frac{3}{x}} = 1$ $\quad$ $\lim_{x\to-\infty} \frac{x-2}{x-3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{1 - \frac{2}{x}}{1 - \frac{3}{x}} = 1$ $\quad$ $\boxed{y = 1 \text{ is H.A.}}$

(d) $f'(x) = \frac{(x-3)\cdot 1 - (x-2)\cdot 1}{(x-3)^2} = \frac{x - 3 - x + 2}{(x-3)^2} = \frac{-1}{(x-3)^2}$ $\quad$ num & den. = 0: $-1 = 0$ no sol $\quad$ $(x-3)^2 = 0$

$\boxed{x = 3 \text{ is critical pt}}$

⑥ continued



$f'(2) = \frac{-1}{(2-3)^2} < 0$ dec

$f'(4) = \frac{-1}{(4-3)^2} < 0$ dec

$\boxed{\begin{array}{l}\text{decreasing: } (-\infty, 3) \cup (3, \infty) \\ \text{increasing: } \emptyset\end{array}}$

* note: no extrema!

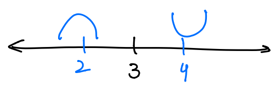
(e) $f'(x) = -1(x-3)^{-2}$

$f''(x) = -1 \cdot (-2)(x-3)^{-3} = \frac{2}{(x-3)^3}$

set num. & den. $= 0$:

$2 = 0$
no sol.

$(x-3)^3 = 0$
$x = 3$



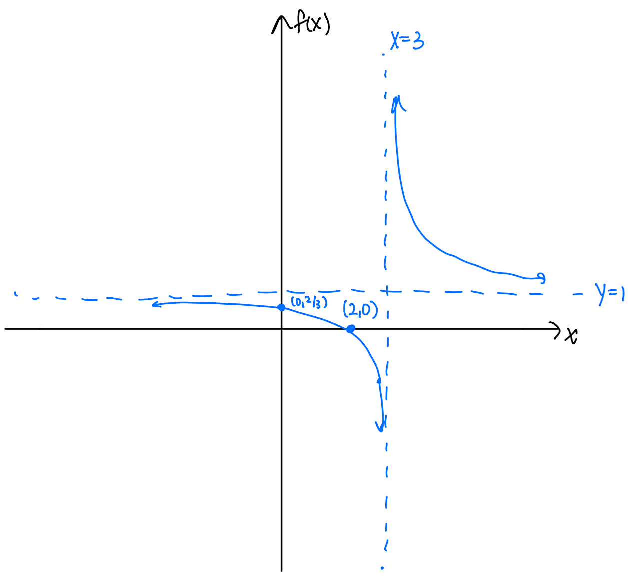
$f''(2) = \frac{2}{(2-3)^3} < 0$ concave down $\cap$

$f''(4) = \frac{2}{(4-3)^3} > 0$ concave up $\cup$

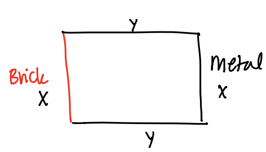
$\boxed{\begin{array}{l}\text{concave up: } (3, \infty) \\ \text{concave down: } (-\infty, 2)\end{array}}$

inflection pt at $x = 3$ $f(3) = \frac{3-2}{3-3}$ undefined $\boxed{\text{no inflection pts}}$

(f)



⑦



Area: $1000 = xy$

$y = \frac{1000}{x}$

Cost: $C = 90x + 30(x + 2y)$

$= 90x + 30x + 60y$

$C = 120x + 60y$

$C = 120x + 60\left(\frac{1000}{x}\right)$

$C = 120x + 60000x^{-1}$

$C' = 120 - 60000x^{-2} = 0$

$\frac{60000}{x^2} = 120$

$120x^2 = 60000$

$x^2 = \frac{6000\cancel{0}}{12\cancel{0}}$

$x^2 = 500$

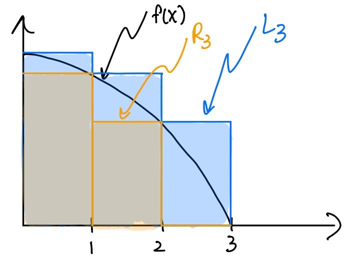
$x = \sqrt{500}$ m

$y = \frac{1000}{\sqrt{500}}$ m

$\boxed{\begin{array}{l}\text{Dimensions are } x = \sqrt{500} \text{ m (brick wall measurement)} \\ y = \frac{1000}{\sqrt{500}} \text{ m}\end{array}}$

⑧ $f(x) = 9 - x^2$

(a), part of (b), part of (c):



(b) $\Delta x = \frac{3-0}{3} = 1$

$L_3 = \Delta x\left(f(0) + f(1) + f(2)\right) = 1\left(9-0^2 + 9-1^2 + 9-2^2\right) = 27-1-4 = 22$ overestimate

(c) $\Delta x = \frac{3-0}{3} = 1$

$R_3 = \Delta x\left(f(1) + f(2) + f(3)\right) = 1\left(9-1^2 + 9-2^2 + 9-3^2\right) = 9-1+9-4 = 18-5 = 13$ underestimate

(d) $\Delta x = \frac{3-0}{N} = \frac{3}{N}$ $\quad x_j = a + j\Delta x = 0 + j\cdot\frac{3}{N} = \frac{3j}{N}$

$$R_N = \frac{3}{N}\sum_{j=1}^{N} f\left(\frac{3j}{N}\right) = \frac{3}{N}\sum_{j=1}^{N}\left(9 - \left(\frac{3j}{N}\right)^2\right) = \frac{3}{N}\sum_{j=1}^{N}\left(9 - \frac{9j^2}{N^2}\right) = \frac{3}{N}\left(\sum_{j=1}^{N} 9 - \sum_{j=1}^{N}\frac{9j^2}{N^2}\right)$$

$$= \frac{3}{N}\left(9\sum_{j=1}^{N} 1 - \frac{9}{N^2}\sum_{j=1}^{N} j^2\right) = \frac{3}{N}\left(9N - \frac{9}{N^2}\cdot\frac{N(N+1)(2N+1)}{6}\right) = \boxed{27 - \frac{9N(N+1)(2N+1)}{2N^3}}$$

(e) $$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(27 - \frac{9N(N+1)(2N+1)}{2N^3}\right) = 27 - \lim_{N\to\infty}\frac{9(N^2+N)(2N+1)}{2N^3} = 27 - \lim_{N\to\infty}\frac{18N^3 + 27N^2 + 9N}{2N^3}$$

$$= 27 - \lim_{N\to\infty}\left(\frac{18N^3}{2N^3} + \frac{27N^2}{3N^3} + \frac{9N}{2N^3}\right) = 27 - 9 = 18$$