Week 6 Solutions

Math 31AL
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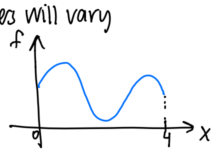
Tuesday, November 5

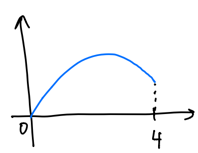
1) $f(x) = x^3 - 2x^2 + 2x - 12$ "increasing" → find the derivative
$f'(x) = 3x^2 - 4x + 2$
Try to find the critical points, i.e. $f'(x) = 0$: $3x^2 - 4x + 2 = 0$
quadratic equation: $x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(2)}}{2(3)} = \frac{4 \pm \sqrt{16-24}}{6} = \frac{4 \pm \sqrt{-8}}{6}$
no real solution, no critical pts

$f'(x)$ doesn't equal 0 → either $f'(x) > 0$ (increasing) or $f'(x) < 0$ (decreasing)
plug in random point: $f'(0) = 2 > 0$, so $f'(x) > 0$ all the time ✓

2) $f(x) = x^3 + 9x - 4$ Intermediate Value Theorem says if ① f continuous ② $f(a) > 0$ ③ $f(b) < 0$ then there is a c between a and b s.t. $f(c) = 0$.

f is a polynomial, so it is continuous
pick a number a s.t. $f(a) > 0$, e.g. $f(1) = 1^3 + 9 \cdot 1 - 4 = 1 + 10 - 4 = 11 - 4 = 7 > 0$
" " " b s.t. $f(b) < 0$. e.g. $f(0) = 0^3 + 9 \cdot 0 - 4 = -4 < 0$
So by IVT, there is a number c between 0 and 1 s.t. $f(c) = 0$ (i.e. a root c) ✓

3) Answers will vary
(a)


(b)


4) $y = 3x^4 + 8x^3 - 6x^2 - 24x$
$y' = 12x^3 + 24x^2 - 12x - 24$ ← plug in 1, get $y' = 12(1)^3 + 24(1)^2 - 12(1) - 24 = 12 + 24 - 12 - 24 = 0$
so $x - 1$ is a factor of $12x^3 + 24x^2 - 12x - 24$
use long division or synthetic division to factor:

$$\begin{array}{r} 12x^2 + 36x + 24 \\ x-1 \overline{\big) 12x^3 + 24x^2 - 12x - 24} \\ -(12x^3 - 12x^2) \qquad\qquad \\ 36x^2 - 12x \qquad \\ -(36x^2 - 36x) \qquad \\ 24x - 24 \\ -(24x - 24) \\ 0 \end{array}$$

so $y' = (x-1)(12x^2 + 36x + 24)$
$= 12(x-1)(x^2 + 3x + 2)$
$= 12(x-1)(x+2)(x+1)$

critical points occur when $y' = 12(x-1)(x+2)(x+1) = 0$
$x - 1 = 0$, $x = 1$
$x + 2 = 0$, $x = -2$
$x + 1 = 0$, $x = -1$

(4) continued

dec inc dec inc

$-3 \quad -2 \quad -1.5 \quad -1 \quad 0 \quad 1 \quad 2$

$$y' = 12(x-1)(x+2)(x+1)$$

Test points, $y'(-3) = 12(-3-1)(-3+2)(-3+1) = 12(-4)(-1)(-2) < 0$ dec

$y'(-1.5) = 12(-1.5-1)(-1.5+2)(-1.5+1) = 12(-2.5)(0.5)(-0.5) > 0$ inc

$y'(0) = 12(0-1)(0+2)(0+1) = 12(-1)(2)(1) < 0$ dec

$y'(2) = 12(2-1)(2+2)(2+1) = 12(1)(4)(3) > 0$ inc

y is increasing on $(-2,-1) \cup (1,\infty)$
decreasing on $(-\infty,-2) \cup (-1,1)$

local max occurs when inc → dec   so local max at $x=-1$
" " " " dec → inc   local mins at $x=-2, 1$

Plug into original function: $y = 3x^4 + 8x^3 - 6x^2 - 24x$

$y(-1) = 3(-1)^4 + 8(-1)^3 - 6(-1)^2 - 24(-1) = 3 - 8 - 6 + 24 = 13$

$y(-2) = 3(-2)^4 + 8(-2)^3 - 6(-2)^2 - 24(-2) = 48 - 64 - 24 + 48 = 8$

$y(1) = 3(1)^4 + 8(1)^3 - 6(1)^2 - 24(1) = 3 + 8 - 6 - 24 = -19$

$(-1,13)$ local max
$(-2,8)$ local min
$(1,-19)$ local min

(5) $f(x) = x^3 + ax + b$

$f'(x) = 3x^2 + a$ want $f'(x) > 0$

$3x^2 \geq 0$ always,

so $f'(x) = 3x^2 + a \geq 0 + a = a$. So we need $a > 0$ for f to be increasing. No restrictions on b.

Thursday, Nov 7

(1) $f(x) = 18(x-3)(x-1)^{2/3}$

Intercepts. x-int, $0 = 18(x-3)(x-1)^{2/3}$

$x - 3 = 0 \qquad (x-1)^{2/3} = 0$

$x = 3 \qquad x = 1$

x-int are $(3,0), (1,0)$

y-int, $f(0) = 18(0-3)(0-1)^{2/3}$

$= 18(-3)(-1)^{2/3}$

$= 18(-3)\sqrt[3]{(-1)^2}$

$= -54 \cdot 1$

$= -54$ y-int is $(0,-54)$

Given $f'(x) = \dfrac{30(x-\frac{9}{5})}{(x-1)^{1/3}}$ Find critical points: $30(x-\frac{9}{5}) = 0$

$x - \frac{9}{5} = 0$

$x = \frac{9}{5}$

$(x-1)^{1/3} = 0$

$x - 1 = 0$

$x = 1$

+ inc   − dec   + inc

$0 \quad 1 \quad 8/5 \quad 9/5 \quad 2$

Test points. $f'(0) = \dfrac{30(0-\frac{9}{5})}{(0-1)^{1/3}} = \dfrac{30(-\frac{9}{5})}{(-1)^{1/3}} = \dfrac{+-}{-} = +$

$f'(\frac{8}{5}) = \dfrac{30(\frac{8}{5}-\frac{9}{5})}{(\frac{8}{5}-1)^{1/3}} = \dfrac{30(-\frac{1}{5})}{(\frac{3}{5})^{1/3}} = \dfrac{+-}{+} = -$

$f'(2) = \dfrac{30(2-\frac{9}{5})}{(2-1)^{1/3}} = \dfrac{30(\frac{10}{5}-\frac{9}{5})}{(1)^{1/3}} = \dfrac{+\cdot+}{+} = +$

inc → dec means local max, local max at $x=1$ $f(1) = 18(1-3)(1-1)^{2/3} = 0$, local max at $(1,0)$

dec → inc means local min, local min at $x = \frac{9}{5}$ $f(\frac{9}{5}) = 18(\frac{9}{5}-3)(\frac{9}{5}-1)^{2/3} \approx -18.6$ local min at $(\frac{9}{5}, -18.6)$

Given $f''(x) = \dfrac{20(x-\frac{3}{5})}{(x-1)^{4/3}}$ set num. & denom. $= 0$: $20(x-\frac{3}{5}) = 0$

$x = \frac{3}{5}$

$(x-1)^{4/3} = 0$

$x = 1$

① continued

$\cap$ $\cup$ $\cup$

$0 \quad 3/5 \quad 4/5 \quad 1 \quad 2$

Test points: $f''(0) = \dfrac{20(0-\frac{3}{5})}{(0-1)^{4/3}} = \dfrac{20(-3/5)}{(-1)^{4/3}} = \dfrac{+\,-}{+} = -$

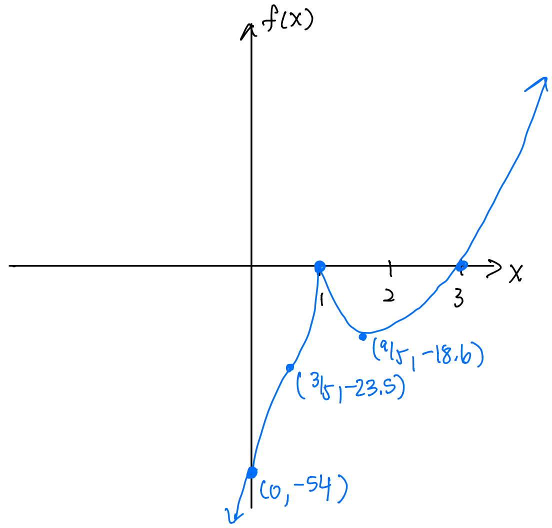
$f''(4/5) = \dfrac{20(\frac{4}{5}-\frac{3}{4})}{(\frac{4}{5}-1)^{4/3}} = \dfrac{20(1/5)}{(-1/5)^{4/3}} = \dfrac{+\,+}{+} = +$

$f''(2) = \dfrac{20(2-3/5)}{(2-1)^{4/3}} = \dfrac{20(\frac{10}{5}-\frac{3}{5})}{1^{4/3}} = \dfrac{+\,+}{+} = +$

$\cap$ to $\cup$ means inflection point at $x = 3/5$ $f(3/5) = 18(\frac{3}{5}-3)(\frac{3}{5}-1)^{1/3} \approx -23.5$ inflection pt at $(3/5, -23.5)$

Asymptotic behavior: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} 18\underbrace{(x-3)}_{\text{big}}\underbrace{(x-1)^{2/3}}_{\text{big}} = \infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} 18\underbrace{(x-3)}_{\text{big} - \#}\underbrace{(x-1)^{2/3}}_{\text{big} + \text{number}} = -\infty$



② on next worksheet