# Week 7 Solutions

Math 31AL
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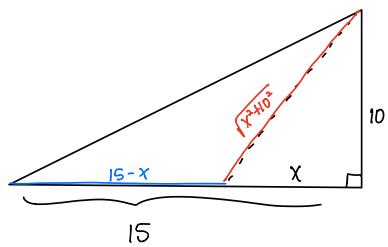
## Tuesday, November 12

(1)




distance on sand: $15-x$

distance in water, use Pythagorean's theorem: $x^2+10^2=c^2$

$\sqrt{x^2+10^2}=c$

Caution: $\sqrt{x^2+10^2} \neq \sqrt{x^2}+\sqrt{10^2}$

Distance = Rate × Time $\longrightarrow$ Time = $\dfrac{\text{Distance}}{\text{Rate}}$

Time on sand: $\dfrac{15-x}{6.4}$ Time in water: $\dfrac{\sqrt{x^2+10^2}}{0.91}$ $\longrightarrow$ Total time: $T(x)=\dfrac{15-x}{6.4}+\dfrac{\sqrt{x^2+100}}{0.91}$

Want to find minimum: $T(x)=\dfrac{1}{6.4}(15-x)+\dfrac{1}{0.91}\cdot(x^2+100)^{1/2}$

$$T'(x)=\frac{1}{6.4}(-1)+\frac{1}{0.91}\cdot\frac{1}{2}(x^2+100)^{-1/2}\cdot 2x=0 \qquad \text{solve for } x$$

$$\frac{-1}{6.4}+\frac{x}{0.91(x^2+100)^{1/2}}=0$$

$$\frac{x}{0.91(x^2+100)^{1/2}}=\frac{1}{6.4} \qquad \text{cross multiply}$$

$$6.4x=0.91(x^2+100)^{1/2}$$

$$\frac{6.4x}{0.91}=(x^2+100)^{1/2} \qquad \text{square both sides}$$

$$\left(\frac{6.4}{0.91}\right)^2x^2=x^2+100$$

$$\left(\frac{6.4}{0.91}\right)^2x^2-x^2=100$$

$$x^2\left(\left(\frac{6.4}{0.91}\right)^2-1\right)=100$$

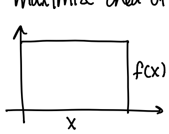
$$x^2=\frac{100}{\left(\frac{6.4}{0.91}\right)^2-1}$$

$$x^2\approx 2.0634$$

$x\approx\pm 1.436$, only positive distance makes sense $x=1.436\text{ m}$

(2) "maximize area of triangle" $\rightarrow$ Find equation for area, $A=xf(x)$

$$A=x(x-2)^2$$




Find max, $A'=1\cdot(x-2)^2+x\cdot 2(x-2)^1\cdot 1=0$

$$(x-2)^2+2x(x-2)=0$$

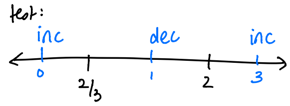
$$(x-2)\left[(x-2)+2x\right]=0$$

② continued

$$(x-2)(3x-2)=0$$

$x-2=0$, $x=2$  $3x-2=0$, $x=2/3$

1st derivative test:




$A'(0)=(0-2)(3\cdot 0-2)>0$ inc

$A'(1)=(1-2)(3\cdot 1-2)<0$ dec

$A'(3)=(3-2)(3\cdot 3-2)>0$ inc

max at $x=2/3$

The point is $(x,f(x))=\left(\frac{2}{3},\left(\frac{2}{3}-2\right)^2\right)$

$=\boxed{\left(\frac{2}{3},\frac{16}{9}\right)}$

③ $f(x)=\dfrac{1}{x^2-10x+24}$

(a) $f(x)=\dfrac{1}{(x-6)(x-4)}$  V.A. when denominator $=0 \to (x-6)(x-4)=0$

$x-6=0$  $x-4=0$

$x=6$  $x=4$  V.A.

H.A. $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{1}{(x-6)(x-4)}=0$  $y=0$ H.A.

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\dfrac{1}{(x-6)(x-4)}=0$

(b) $f(x)=\dfrac{1}{(x-6)(x-4)}$  quotient rule: $f'(x)=\dfrac{\cancel{(x-6)(x-4)}(1)'-1((x-6)(x-4))'}{((x-6)(x-4))^2}$

$f'(x)=\dfrac{-\left((x-6)'(x-4)+(x-6)(x-4)'\right)}{(x-6)^2(x-4)^2}$

$=\dfrac{-(x-4+x-6)}{(x-6)^2(x-4)^2}$

$\boxed{f'(x)=\dfrac{-(2x-10)}{(x-6)^2(x-4)^2}}$

set numerator & denominator $=0$

$-(2x-10)=0$

$2x-10=0$

$2x=10$

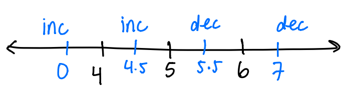
$x=5$

$(x-6)^2(x-4)^2=0$

$(x-6)^2=0$  $(x-4)^2=0$

$x-6=0$  $x-4=0$

$x=6$  $x=4$




$f'(0)=-\dfrac{(2\cdot 0-10)}{(0-6)^2(0-4)^2}=\dfrac{10}{6^2 4^2}>0$ inc

$f'(4.5)=-\dfrac{(2(4.5)-10)}{(4.5-6)^2(4.5-4)^2}=\dfrac{-(9-10)}{(-1.5)^2(0.5)^2}>0$ inc

$f'(5.5)=\dfrac{-(2(5.5)-10)}{(5.5-6)^2(5.5-4)^2}=\dfrac{-(11-10)}{(0.5)^2(1.5)^2}<0$ dec

$f'(7)=\dfrac{-(2(7)-10)}{(7-6)^2(7-4)^2}=\dfrac{-(14-10)}{1^2\cdot 3^2}<0$ dec

local max at $x=5$

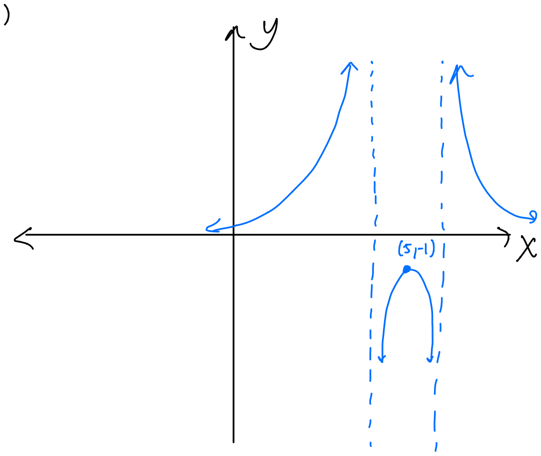
$f(5)=\dfrac{1}{(5-6)(5-4)}=\dfrac{1}{(-1)(1)}=-1$,

$\boxed{\text{local max at } (5,-1)}$

(c) see (d)

③ continued

(d)




(e) $f'(x) = -\dfrac{(2x-10)}{(x-6)^2(x-4)^2}$ based on the graph above, expect concave down on $(4,6)$, concave up on $(-\infty,4)\cup(6,\infty)$

$$f''(x) = \frac{(x-6)^2(x-4)^2\left[-(2x-10)\right]' - \left[-(2x-10)\right]\left((x-6)^2(x-4)^2\right)'}{\left((x-6)^2(x-4)^2\right)^2}$$

$$= \frac{(x-6)^2(x-4)^2(-2) + (2x-10)\left(2(x-6)(x-4)^2 + (x-6)^2\cdot 2(x-4)\right)}{(x-6)^4(x-4)^4}$$

$$= \frac{-2(x-6)^2(x-4)^2 + 2(2x-10)(x-6)(x-4)(x-4+x-6)}{(x-6)^4(x-4)^4}$$

$$= \frac{(x-6)(x-4)\left[-2(x-6)(x-4) + 2(2x-10)(2x-10)\right]}{(x-6)^4(x-4)^4}$$

$$= \frac{-2(x^2-10x+24) + 2(4x^2-40x+100)}{(x-6)^3(x-4)^3}$$

$$= \frac{6x^2+60x+152}{(x-6)^3(x-4)^3}$$

set numer. & denom. $= 0$:

$6x^2+60x+152=0$

$3x^2-30x+76=0$

$x = \dfrac{30 \pm \sqrt{(-30)^2-4(3)(76)}}{2(3)}$

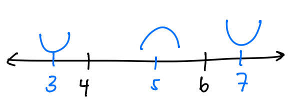
$= \dfrac{30 \pm \sqrt{900-912}}{6}$

no real solutions

$(x-6)^3(x-4)^3=0$

$x-6=0$, $x=6$

$x-4=0$, $x=4$




$f''(3) = \dfrac{6(3)^2+60(3)+152}{(3-6)^3(3-4)^3} = \dfrac{+}{-\cdot-} > 0$ concave up

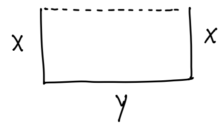
$f''(5) = \dfrac{6(5)^2+60(5)+152}{(5-6)^3(5-4)^3} = \dfrac{+}{-\,+} < 0$ concave down

$f''(7) = \dfrac{6(7)^3+60(7)+152}{(7-6)^3(7-4)^3} = \dfrac{+}{++} > 0$ concave up

$\boxed{\text{Concave up: } (-\infty,4)\cup(6,\infty) \quad \text{concave down: } (4,6)}$

Thursday, November 14

① (a)




"Maximize area" — Find equation for area

$A = xy$ ← narrow down to 1 variable

Perimeter: $40 = 2x + y$

$y = 40 - 2x$

$A = x(40-2x)$

$A = 40x - 2x^2$

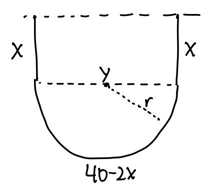
$A' = 40 - 4x = 0$

$40 = 4x$

$x = 10$, so $y = 40 - 2(10) = 20$ $\boxed{\begin{array}{l} x = 10\,ft \\ y = 20\,ft \end{array}}$

(the area would be 200 ft)

(b)




$A = xy + \frac{1}{2}r^2$ need to narrow down to 1 variable

$y = 2r \rightarrow A = x(2r) + \frac{1}{2}r^2$

$A = 2xr + \frac{1}{2}r^2$

circumference: $40 - 2x = \cancel{\frac{1}{2}} \cdot \cancel{2}\pi r$

$r = \frac{40-2x}{\pi}$

$A = 2x\left(\frac{40-2x}{\pi}\right) + \frac{1}{2}\left(\frac{40-2x}{\pi}\right)^2$

$= \frac{80x - 4x^2}{\pi} + \frac{1}{2}\left(\frac{40-2x}{\pi}\right)^2$

$A' = \frac{1}{\pi}(80 - 8x) + \frac{1}{\cancel{2}} \cdot \cancel{2}\left(\frac{40-2x}{\pi}\right) \cdot \left(\frac{-2}{\pi}\right) = 0$

$\frac{80-8x}{\pi} - 2\left(\frac{40-2x}{\pi^2}\right) = 0$

$\frac{80-8x}{\pi} = \frac{2(40-2x)}{\pi^2}$ cross multiply

$\pi^{\cancel{2}}(80-8x) = 2\cancel{\pi}(40-2x)$

$80\pi - 8\pi x = 80 - 4x$

$80\pi - 80 = 8\pi x + 4x$

$80\pi - 80 = (8\pi + 4)x$

$x = \frac{80\pi - 80}{8\pi + 4} \approx 5.88$ ft

$A = \frac{80(5.88) - 4(5.88)^2}{\pi} + \frac{1}{2}\left(\frac{40 - 2(5.88)}{\pi}\right)^2$

$\approx 146.113$ ft

not bigger than (a)

② (i) V.A. $x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1 \rightarrow$ C or D

H.A. $\lim_{x\to\infty} \frac{1}{x^2-1} = 0 \rightarrow$ still C or D

at $x = 0$, $\frac{1}{0^2-1} = \frac{1}{-1} = -1$, → D

② continued

(ii) V.A. $x^2-1=0$ but if still not sure, H.A. $\lim_{x\to\infty} \frac{x}{x^2-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{x-\frac{1}{x}} = 0$

$x=\pm 1 \rightarrow$ C or D, C

at $x=0$, $\frac{0}{0^2-1} = 0.$

(iii) V.A. $x^2+1=0$ H.A. $\lim_{x\to\infty} \frac{1}{x^2+1} = 0 \rightarrow$ B

$x^2=-1$

no solution

no V.A. $\longrightarrow$ A or B

(iv) V.A. $x^2+1=0$ H.A. $\lim_{x\to\infty} \frac{x^2}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1 \rightarrow$ A

no solution

no V.A.

③ $f(x) = (x-1)^2(x-2)(x-3)$

(a) product rule:

$$\begin{aligned} f'(x) &= ((x-1)^2)'(x-2)(x-3) + (x-1)^2(x-2)'(x-3) + (x-1)^2(x-2)(x-3)' \\ &= 2(x-1)(x-2)(x-3) + (x-1)^2\cdot 1\cdot(x-3) + (x-1)^2(x-2)\cdot 1 \\ &= (x-1)\left[2(x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)\right] \\ &= (x-1)\left[2(x^2-5x+6) + x^2-4x+3 + x^2-3x+2\right] \\ &= (x-1)\left[2x^2-10x+12 + 2x^2-7x+5\right] \\ 0 &= (x-1)\left[4x^2-17x+17\right] \end{aligned}$$

$x-1=0$

$x=1$

$4x^2-17x+17=0$

$x = \frac{17\pm\sqrt{17^2-4(4)(17)}}{2(4)} = \frac{17\pm\sqrt{289-272}}{8} = \frac{17\pm\sqrt{17}}{8}$

(b)

$$\begin{aligned} f'(x) &= (x-1)(4x^2-17x+17) \\ f''(x) &= (x-1)'(4x^2-17x+17) + (x-1)(4x^2-17x+17)' \\ &= 4x^2-17x+17+(x-1)(8x-17) \\ &= 4x^2-17x+17 + 8x^2-17x-8x+17 \\ &= 12x^2-42x+34 \end{aligned}$$

$f''(1) = 12-42+34 > 0$ concave up $\cup$ min at $x=1$

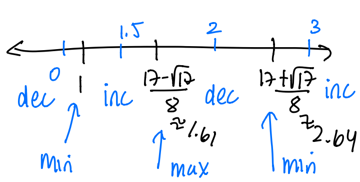
$f''\left(\frac{17+\sqrt{17}}{8}\right) \approx f''(2.64) = 12(2.64)^2-42(2.64)+34$

$= 6.7552 > 0$ concave up min at $x=\frac{17+\sqrt{17}}{8}$

$f''\left(\frac{17-\sqrt{17}}{8}\right) \approx f''(1.61) = 12(1.61)^2-42(1.61)+34 = -2.5148 < 0$

concave down max at $x=\frac{17-\sqrt{17}}{8}$

(c)




$f' = (x-1)(4x^2-17x+17)$

$f'(0) = (-1)(17) < 0$ dec

$f'(1.5) = (1.5-1)(4(1.5)^2-17(1.5)+17) > 0$ inc

$f'(2) = (2-1)(4(2)^2-17(2)+17) < 0$ dec

$f'(3) = (3-1)(4(3)^2-17(3)+17) > 0$ inc

local min at $x=1, \frac{17+\sqrt{17}}{8}$

local max at $x=\frac{17-\sqrt{17}}{8}$

③ continued

(d) $f(1) = (1-1)^2(1-2)(1-3) = 0$

$f\left(\frac{17+\sqrt{17}}{8}\right) \approx f(2.64) = (2.64-1)^2(2.64-2)(2.64-3) \approx -0.62$

$f\left(\frac{17-\sqrt{17}}{8}\right) \approx f(1.61) = (1.61-1)^2(1.61-2)(1.61-3) \approx 0.2$

intercepts: $(x-1)^2(x-2)(x-3) = 0$

$x = 1, 2, 3$

$(1,0), (2,0), (3,0)$

$f(0) = (0-1)^2(0-2)(0-3) = 6$

$(0,6)$