Week 8 Solutions

Math 31AL
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Tuesday, November 19

① $3x^2+4y^2+3xy=24$ using implicit differentiation: $6x+8yy'+\underbrace{3y+3xy'}_{\text{product rule}}=0$

$$8yy'+3xy'=-6x-3y$$
$$y'(8y+3x)=-6x-3y$$
$$y'=\frac{-6x-3y}{8y+3x}$$

Want $y'=0 \Rightarrow \frac{-6x-3y}{8y+3x}=0$

$-6x-3y=0$ solve for either x or y

$3y=-6x$

$y=-2x$

Plug into original: $3x^2+4y^2+3xy=24$

$3x^2+4(-2x)^2+3x(-2x)=24$

$3x^2+16x^2-6x^2=24$

$13x^2=24$

$x^2=\frac{24}{13}$

$x=\pm\sqrt{\frac{24}{13}}\approx 1.36$

If $x=\sqrt{\frac{24}{13}}$, then $y=-2\left(\sqrt{\frac{24}{13}}\right)\approx -2.717$

If $x=-\sqrt{\frac{24}{13}}$, then $y=-2\left(-\sqrt{\frac{24}{13}}\right)\approx 2.717$

$\boxed{\text{The points are } \left(\sqrt{\tfrac{24}{13}},-2\sqrt{\tfrac{24}{13}}\right)\approx(1.36,-2.72) \text{ and } \left(-\sqrt{\tfrac{24}{13}},2\sqrt{\tfrac{24}{13}}\right)\approx(-1.36,2.72)}$

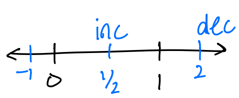
② (a) $f(x)=2\sqrt{x}-x=2x^{1/2}-x$

$f'(x)=2\cdot\frac{1}{2}x^{-1/2}-1=\frac{1}{\sqrt{x}}-1=\frac{1}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{x}}=\frac{1-\sqrt{x}}{\sqrt{x}}=0$

common denominator

$1-\sqrt{x}=0\to\sqrt{x}=1\to x=1$

$\sqrt{x}=0\to x=0$



Test: $f'(-1)$ undefined because of $\sqrt{\ }$

$f'(\frac{1}{2})=\frac{1-\sqrt{1/2}}{\sqrt{1/2}}>0$ inc

$f'(2)=\frac{1-\sqrt{2}}{\sqrt{2}}<0$ dec

$x=1$ is local max

$f(1)=2\sqrt{1}-1=2\cdot1-1=1$

$f(0)=2\sqrt{0}-0=0$

$f(4)=2\sqrt{4}-4=2(2)-4=0$

$\boxed{\begin{array}{l}(1,1) \text{ is absolute max}\\ (0,0) \text{ and } (4,0) \text{ are abs. mins}\end{array}}$

(2) continued

(b) $f(x) = 2x^3 + 3x^2 - 36x - 2$

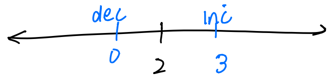
$f'(x) = 6x^2 + 6x - 36 = 0$

$6(x^2 + x - 6) = 0$

$6(x+3)(x-2) = 0$

$\cancel{x=-3}$, $x = 2$

not in [0,4]



$f'(0) = 6(0+3)(0-2) < 0$ dec

$f'(3) = 6(3+3)(3-2) > 0$ inc

$x=2$ is local min

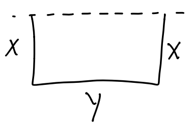
$f(2) = 2(2)^3 + 3(2)^2 - 36(2) - 2 = 16 + 12 - 72 - 2 = -46$

$f(0) = 2(0)^3 + 3(0)^2 - 36(0) - 2 = -2$

$f(4) = 2(4)^3 + 3(4)^2 - 36(4) - 2 = 128 + 48 - 144 - 2 = 30$

$\boxed{\begin{array}{l}\text{abs. max at } (4,30) \\ \text{abs. min at } (2,-46)\end{array}}$

(3)



"maximize area" → need formula for area $A = xy$

given 400ft fencing: $400 = 2x + y$

$y = 400 - 2x$

$A = x(400 - 2x)$

$A = 400x - 2x^2$

$A' = 400 - 4x = 0$

$4x = 400$

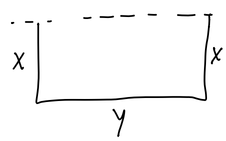
$x = 100$ ft

$y = 400 - 2(100)$

$y = 200$ ft

$A = xy = 100(200)\ ft^2 = 20000\ ft^2$

(4) Not given pricing or info about materials...



Suppose p = price of fencing materials

(if x, y are made from same material)

$C = p(2x + y)$

given $100 = xy \rightarrow y = \frac{100}{x}$

$C = p\left(2x + \frac{100}{x}\right) = p\left(2x + 100x^{-1}\right)$

$C' = p\left(2 - 100x^{-2}\right) = 0$

$2 - \frac{100}{x^2} = 0 \rightarrow 2 = \frac{100}{x^2} \rightarrow 2x^2 = 100 \rightarrow x^2 = 50 \rightarrow x = \sqrt{50}$ ft

$y = \frac{100}{\sqrt{50}}$ ft

Another way:

If x, y are different materials then let a = cost of x material

b = " " y material

$C = a \cdot 2x + b \cdot y$

given $100 = xy \rightarrow y = \frac{100}{x}$

$= 2ax + \frac{100b}{x} = 2ax + 100bx^{-1}$

$C' = 2a - 100bx^{-2} = 0$

④ continued

$$2a - \frac{100b}{x^2} = 0 \rightarrow \frac{100b}{x^2} = 2a \rightarrow 100b = 2ax^2 \rightarrow x^2 = \frac{100b}{2a} \rightarrow x^2 = \frac{50b}{a}$$

$$\boxed{x = \sqrt{\frac{50b}{a}}\ \text{ft}}$$

$$y = \frac{100}{x} = \frac{100}{\sqrt{\frac{50b}{a}}} = \boxed{100\sqrt{\frac{a}{50b}}\ \text{ft}}$$

Thursday, November 21

① intercepts x-int: $0 = \frac{2x^2-1}{x^2+1}$

$0 = 2x^2 - 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

$\boxed{\text{x-int}: \left(\sqrt{\tfrac{1}{2}}, 0\right), \left(-\sqrt{\tfrac{1}{2}}, 0\right)}$

y-int: $f(0) = \frac{2(0)^2-1}{0^2+1} = \frac{-1}{1} = -1$

$\boxed{\text{y-int } (0,-1)}$

asymptotes: den. = 0 $\quad x^2+1=0$

$x^2 = -1$

no sol. $\boxed{\text{no V.A.}}$

$$\lim_{x\to\infty} \frac{2x^2-1}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2-\frac{1}{x^2}}{1+\frac{1}{x^2}} = 2$$

$$\lim_{x\to-\infty} \frac{2x^2-1}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{2-\frac{1}{x^2}}{1+\frac{1}{x^2}} = 2$$

$\boxed{\text{H.A. } y=2}$

$$f'(x) = \frac{(x^2+1)\cdot 4x - (2x^2-1)\cdot 2x}{(x^2+1)^2} = \frac{4x^3+4x-4x^3+2x}{(x^2+1)^2} = \frac{6x}{(x^2+1)^2} = 0$$

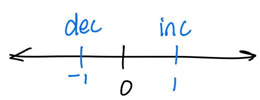
$6x = 0$

$x = 0$ ← critical pt.

$(x^2+1)^2 = 0$

$x^2+1 = 0$

no sol.



$f'(-1) = \frac{6(-1)}{((-1)^2+1)^2} < 0$ dec $\qquad f'(1) = \frac{6\cdot 1}{(1^2+1)^2} > 0$ inc

local min at $x=0$, $f(0) = \frac{-1}{1} = -1$ $\qquad \boxed{\text{local min at } (0,-1)}$

$$f'(x) = \frac{6x}{(x^2+1)^2} \qquad f''(x) = \frac{(x^2+1)^2\cdot 6 - 6x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{(x^2+1)\left[6-24x^2\right]}{(x^2+1)^4} = \frac{6-24x^2}{(x^2+1)^3}$$

$6 - 24x^2 = 0$

$24x^2 = 6$

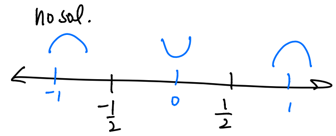
$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

$(x^2+1)^3 = 0$

$x^2+1 = 0$

no sol.



$f''(-1) = \frac{6-24(-1)^2}{((-1)^2+1)^3} < 0$ concave down ∩

$f''(0) = \frac{6-24(0)^2}{((0)^2+1)^3} > 0$ concave up ∪

$f''(1) = \frac{6-24\cdot 1^2}{(1^2+1)^3} < 0$ concave down ∩

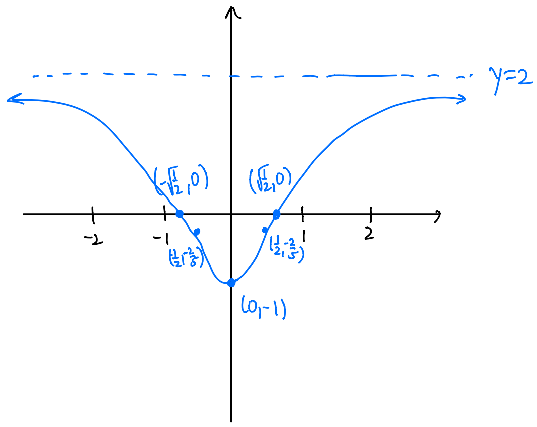
$x = \pm\frac{1}{2}$ are inflection pts,

$$f\left(\tfrac{1}{2}\right) = \frac{2\left(\frac{1}{2}\right)^2-1}{\left(\frac{1}{2}\right)^2+1} = \frac{\frac{2}{4}-1}{\frac{1}{4}+1} = \frac{-\frac{1}{2}}{\frac{5}{4}} = -\frac{1}{2}\cdot\frac{4}{5} = -\frac{2}{5}$$

$$f\left(-\tfrac{1}{2}\right) = \frac{2\left(-\frac{1}{2}\right)^2-1}{\left(-\frac{1}{2}\right)^2+1} = \frac{\frac{2}{4}-1}{\frac{1}{4}+1} = \dots = -\frac{2}{5}$$

$\boxed{\text{inflection pts } \left(\tfrac{1}{2}, -\tfrac{2}{5}\right), \left(-\tfrac{1}{2}, -\tfrac{2}{5}\right)}$

① continued



② Volume $= 400\,m^3$

$\pi r^2 h = 400$

$h = \dfrac{400}{\pi r^2}$

Cost: $C = 40(2\pi r^2) + 60(2\pi r h)$

$C = 80\pi r^2 + 120\pi r h$

$C = 80\pi r^2 + 120\cancel{\pi}\cancel{r} \cdot \dfrac{400}{\cancel{\pi r^2}}$

$C = 80\pi r^2 + 48000 r^{-1}$

$C' = 160\pi r - 48000 r^{-2} = 0$

$160\pi r = \dfrac{48000}{r^2}$

$160\pi r^3 = 48000$

$r^3 = \dfrac{48000}{160\pi}$

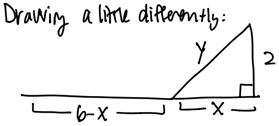
$r = \sqrt[3]{\dfrac{48000}{160\pi}} \approx 4.57\,m$

③ $D = RT \rightarrow T = \dfrac{D}{R}$

Time = time running + time swimming

= $\dfrac{\text{distance running}}{\text{rate running}} + \dfrac{\text{distance swimming}}{\text{rate swimming}}$

Drawing a little differently:



$y = \sqrt{x^2+2^2}$

$= \sqrt{x^2+4}$

$T(x) = \dfrac{6-x}{8} + \dfrac{\sqrt{x^2+4}}{3} = \dfrac{6-x}{8} + \dfrac{(x^2+4)^{1/2}}{3}$

$T'(x) = -\dfrac{1}{8} + \dfrac{\frac{1}{\cancel{2}}(x^2+4)^{-1/2} \cdot \cancel{2}x}{3} = 0$

$\dfrac{1}{8} = \dfrac{x}{3\sqrt{x^2+4}} \rightarrow \left(3\sqrt{x^2+4}\right)^2 = (8x)^2 \rightarrow 9(x^2+4) = 64x^2$

$9x^2+36 = 64x^2$

$36 = 55x^2$

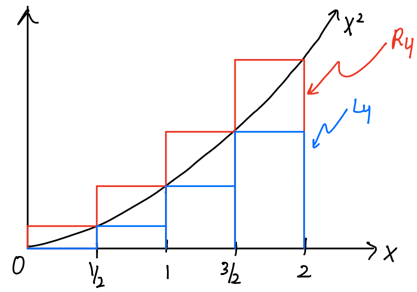
$x^2 = \dfrac{55}{36} \rightarrow x = \pm\sqrt{\dfrac{55}{6}}$

only need positive

$x = \sqrt{\dfrac{55}{6}} \approx 3.03$ mi

The visitor should run $6 - \sqrt{\frac{55}{6}} \approx 2.97$ mi

④



(a) $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

$L_4 = \Delta x\left(f(0) + f(1/2) + f(1) + f(3/2)\right)$
$= \frac{1}{2}\left(0^2 + \left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{3}{2}\right)^2\right)$
$= \frac{1}{2}\left(0 + \frac{1}{4} + 1 + \frac{9}{4}\right) = \frac{1}{2}\left(\frac{10}{4} + 1\right) = \frac{1}{2}\left(\frac{5}{2} + 1\right) = \frac{7}{4}$

(b) $R_4 = \Delta x\left(f(\frac{1}{2}) + f(1) + f(\frac{3}{2}) + f(2)\right) = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{3}{2}\right)^2 + 2^2\right) = \frac{1}{2}\left(\frac{1}{4} + 1 + \frac{9}{4} + 4\right) = \frac{1}{2}\left(\frac{10}{4} + 5\right) = \frac{1}{2}\left(\frac{5}{2} + 5\right)$
$= \frac{1}{2}\left(\frac{15}{2}\right) = \frac{15}{4}$

(c) The exact area is between $\frac{7}{4}$ and $\frac{15}{4}$.

If the problem asked for the exact area, use $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ and $R_n = \Delta x \sum_{i=0}^{n} f(x_i)$

In our case, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_i = 0 + i\frac{2}{n} = \frac{2i}{n}$

$$R_n = \Delta x \sum_{i=0}^{n} f(x_i) = \frac{2}{n}\sum_{i=0}^{n} f\left(\frac{2i}{n}\right) = \frac{2}{n}\sum_{i=0}^{n}\left(\frac{2i}{n}\right)^2 = \frac{2}{n}\sum_{i=0}^{n}\frac{4i^2}{n^2} = \frac{2}{n}\cdot\frac{4}{n^2}\sum_{i=0}^{n} i^2 = \frac{8}{n^3}\sum_{i=0}^{n} i^2$$

$$= \frac{8}{\cancel{n^3}\,n^2}\,\frac{\cancel{n}(n+1)(2n+1)}{6} = \frac{8(2n^2+n+2n+1)}{6n^2} = \frac{16n^2+24n+8}{6n^2}$$

$$\text{exact area} = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{16n^2+24n+8}{6n^2} = \lim_{n\to\infty}\frac{16n^2}{6n^2} + \frac{24n}{6n^2} + \frac{8}{6n^2} = \lim_{n\to\infty}\frac{16}{6} + \frac{24}{6n} + \frac{8}{6n^2} = \frac{16}{6} = \frac{8}{3}$$