Week 9 Solutions

Math 31AL
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① An antiderivative of $f(x)$ is $\int f(x)\,dx = F(x) + C$ (indefinite integral)

Area under $f(x)$ on $[a,b]$ is $\int_a^b f(x)\,dx = F(b) - F(a)$ (definite integral)

② $f(x) = \int (x^3 - 3x^2 + 7)\,dx = \dfrac{x^4}{4} - x^3 + 7x + C$

when $x=1$, $f=1$:
$$1 = \frac{1^4}{4} - 1^3 + 7\cdot 1 + C$$
$$1 = \frac{1}{4} - 1 + 7 + C$$
$$1 = \frac{1}{4} + 6 + C$$
$$1 = \frac{25}{4} + C$$
$$1 - \frac{25}{4} = C$$
$$-\frac{21}{4} = C$$

$$\boxed{f(x) = \frac{x^4}{4} - x^3 + 7x - \frac{21}{4}}$$

③ $f'(x) = \int \sin(3x)\,dx = \int \sin u \,\dfrac{du}{3} = \dfrac{1}{3}\int \sin u\,du = -\dfrac{1}{3}\cos u + C = -\dfrac{1}{3}\cos(3x) + C$

$u = 3x$ $\quad du = 3\,dx$ $\quad dx = \dfrac{du}{3}$

$f'(0) = 1$ means $x=0$, $f'=1$: $\quad 1 = -\frac{1}{3}\cos(3\cdot 0) + C \rightarrow 1 = -\frac{1}{3}\cos 0 + C \rightarrow 1 = -\frac{1}{3} + C \rightarrow C = \frac{4}{3}$

$$f(x) = \int\left(-\frac{1}{3}\cos(3x) + \frac{4}{3}\right)dx = \int\left(-\frac{1}{3}\cos(u) + \frac{4}{3}\right)\cdot\frac{du}{3} = \frac{1}{3}\int\left(-\frac{1}{3}\cos u + \frac{4}{3}\right)du = \frac{1}{3}\left(-\frac{1}{3}\sin u + \frac{4}{3}u\right) + D$$

$u = 3x$ $\quad du = 3\,dx$ $\quad dx = \dfrac{du}{3}$

$$= \frac{1}{3}\left(-\frac{1}{3}\sin(3x) + \frac{4}{\cancel{3}}\cdot 3x\right) + D = -\frac{1}{9}\sin 3x + \frac{4}{3}x + D$$

$f(0) = 0$ means $f=0$, $x=0$: $\quad 0 = -\frac{1}{3}\sin(3\cdot 0) + \frac{4}{3}\cdot 0 + D \rightarrow 0 = D \rightarrow \boxed{f(x) = -\frac{1}{3}\sin 3x + \frac{4}{3}x}$

④ $v(t) = \sin(t)$

$s(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C$

But $s=0$ when $t=0$:
$$0 = -\cos 0 + C$$
$$0 = -1 + C$$
$$C = 1$$

$$\boxed{s(t) = -\cos t + 1}$$

⑤ $\dfrac{dv}{dr} = -0.06r \Rightarrow v = \int -0.06r\,dr = -0.06\dfrac{r^2}{2} + C = -0.03r^2 + C$

when $r=10$, $v=0$: $\quad 0 = -0.03(10)^2 + C \rightarrow 0 = -0.03(100) + C \rightarrow C = 3 \rightarrow \boxed{v = -0.03r^2 + 3}$

6 (a) $\int x^5\,dx = \boxed{\frac{x^6}{6}+C}$ Check: $\left(\frac{x^6}{6}\right)' = \frac{6x^5}{6} = x^5$ ✓

(b) $\int \frac{1}{x^3}\,dx = \int x^{-3}dx = \boxed{\frac{x^{-2}}{-2}+C}$ Check: $\left(\frac{x^{-2}}{-2}\right)' = \frac{-2x^{-3}}{-2} = x^{-3}$ ✓

(c) $\int \frac{1}{(2x+1)^2}\,dx$ $u=2x+1$, $du=2dx \longrightarrow dx=\frac{du}{2}$ $\int \frac{1}{u^2}\,\frac{du}{2} = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\,\frac{u^{-1}}{-1} = \boxed{\frac{-1}{2}(2x+1)^{-1}}$

Check: $\left(\frac{-1}{2}(2x+1)^{-1}+C\right)' = \frac{-1}{\cancel{2}}\cdot -1(2x+1)^{-2}\cdot \cancel{2} = (2x+1)^{-2}$ ✓

(d) $\int \frac{t^2+1}{\sqrt{t}}\,dt = \int \frac{t^2}{\sqrt{t}}dt + \int \frac{1}{\sqrt{t}}dt = \int \frac{t^2}{t^{1/2}}dt + \int \frac{1}{t^{1/2}}dt = \int t^{3/2}dt + \int t^{-1/2}dt = \frac{t^{5/2}}{5/2} + \frac{t^{1/2}}{1/2} + C$

$= \boxed{\frac{2}{5}t^{5/2} + 2t^{1/2} + C}$

Check: $\left(\frac{2}{5}t^{5/2} + 2t^{1/2} + C\right)' = \frac{\cancel{2}}{\cancel{5}}\cdot\frac{\cancel{5}}{\cancel{2}}t^{3/2} + 2\cdot\frac{1}{2}t^{-1/2} = t^{3/2} + t^{-1/2}$ ✓

(e) $\int \cos(3x)\,dx$ $u=3x$, $du=3dx \rightarrow dx=\frac{du}{3}$ $\int \cos u\,\frac{du}{3} = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \boxed{\frac{1}{3}\sin(3x)+C}$

check: $\left(\frac{1}{3}\sin(3x)+C\right)' = \frac{1}{\cancel{3}}\cos(3x)\cdot\cancel{3} = \cos(3x)$ ✓

(f) $\int \cos x\cdot 3\sin^2 x\,dx = \int \cos x\cdot 3(\sin x)^2dx$ $u=\sin x$, $du=\cos x\,dx$ $\int 3u^2du = u^3+C = \boxed{(\sin x)^3+C}$

Check: $\left((\sin x)^3\right)' = 3(\sin x)^2\cos x$ ✓

(g) $\int 2x\sec^2(x^2)\,dx$ $u=x^2$, $du=2x\,dx$ $\int \sec^2 u\,du = \tan u + C = \boxed{\tan(x^2)+C}$ Check: $\left(\tan(x^2)+C\right)' = \sec^2(x^2)\cdot 2x$ ✓

7 (a) $\int_0^3 x^{1/2}\,dx = \frac{x^{3/2}}{3/2}\Big|_0^3 = \frac{2}{3}x^{3/2}\Big|_0^3 = \frac{2}{3}\left(3^{3/2}-0^{3/2}\right) = \frac{2}{3}\,3^{3/2} = \frac{2}{\cancel{3}}\cdot\cancel{3}\sqrt{3} = 2\sqrt{3}$

(b) $\int_{-3}^5 x(x^{2/3}+1)\,dx = \int_{-3}^5 (x^{5/3}+x)\,dx = \left(\frac{x^{8/3}}{8/3}+\frac{x^2}{2}\right)\Big|_{-3}^5 = \left(\frac{3}{8}x^{8/3}+\frac{x^2}{2}\right)\Big|_{-3}^5 = \frac{3}{8}\cdot 5^{8/3}+\frac{5^2}{2}-\left(\frac{3}{8}(-3)^{8/3}+\frac{(-3)^2}{2}\right)$

$= \frac{3}{8}\cdot 5^{8/3}+\frac{25}{2}-\frac{3}{8}(-3)^{8/3}-\frac{9}{2} = \boxed{\frac{3}{8}\left(5^{8/3}-(-3)^{8/3}\right)+8}$

(c) $\int_{-3}^5 |x(x^{2/3}+1)|\,dx = \int_{-3}^0 |x(x^{2/3}+1)|dx + \int_0^5 |x(x^{2/3}+1)|\,dx = \int_{-3}^0 -x(x^{2/3}+1)dx + \int_0^5 x(x^{2/3}+1)dx$

* use $|x| = \begin{cases} -x & x<0 \\ x & x\geq 0 \end{cases}$

$= \int_{-3}^0 (-x^{5/3}-x)dx + \int_0^5 (x^{5/3}+x)dx$

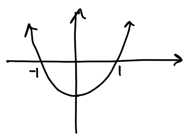
$= \left(\frac{-3}{8}x^{8/3}-\frac{x^2}{2}\right)\Big|_{-3}^0 + \left(\frac{3}{8}x^{8/3}+\frac{x^2}{2}\right)\Big|_0^5$

$= \left(\cancel{\frac{-3\cdot 0^{8/3}}{8}-\frac{0^2}{2}}\right) - \left(\frac{-3(-3)^{8/3}}{8}-\frac{(-3)^2}{2}\right) + \frac{3}{8}\cdot 5^{8/3}+\frac{5^2}{2}-\left(\cancel{\frac{3}{8}\cdot 0^{8/3}}+\frac{0^2}{2}\right) = \frac{3(-3)^{8/3}}{8}+\frac{9}{2}+\frac{3}{8}\cdot 5^{8/3}+\frac{25}{2}$

$= \boxed{\frac{3}{8}\left((-3)^{8/3}+5^{8/3}\right)+\frac{34}{2}}$

(d) $\int_0^{\pi/4} \sec^2\theta\,d\theta = \tan\theta\Big|_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right)-\tan 0 = 1-0 = 1$

(e) $\int_{-5}^5 |t^2-1|\,dt = \int_{-5}^{-1}(t^2-1)dt + \int_{-1}^1 -(t^2-1)\,dt + \int_1^5 (t^2-1)dt = \left(\frac{t^3}{3}-t\right)\Big|_{-5}^{-1} - \left(\frac{t^3}{3}-t\right)\Big|_{-1}^1 + \left(\frac{t^3}{3}-t\right)\Big|_1^5$



$= \frac{(-1)^3}{3}-(-1)-\left(\frac{(-5)^3}{3}-(-5)\right)-\left(\left(\frac{1^3}{3}-1\right)-\left(\frac{(-1)^3}{3}-(-1)\right)\right)+\frac{5^3}{3}-5-\left(\frac{1^3}{3}-1\right)$

$= \frac{-1}{3}+1-\left(\frac{-125}{3}+5\right)-\left(\frac{1}{3}-1-\left(1-\frac{1}{3}\right)\right)+\frac{125}{3}-5-\left(\frac{1}{3}-1\right) = \frac{2}{3}+\frac{125}{3}-5-\left(\frac{-2}{3}-\frac{2}{3}\right)+\frac{125}{3}-5+\frac{2}{3} = \frac{8}{3}+\frac{250}{3}-10$

$= \frac{258}{3}-10 = \frac{228}{3}$