Math 33B: Method of Undetermined Coefficients, Eigenvalues and Eigenvectors

Written by Victoria Kala

November 5, 2019

Name: KEY

1. Solve $y'' + 2y' - 3y = 5\sin 3t$. (*Hint:* Try a particular solution of the form $y_p = A\cos 3t + B\sin 3t$.)

① homogeneous

$y'' + 2y' - 3y = 0$

$m^2 + 2m - 3 = 0$

$(m+3)(m-1) = 0$

$m = -3 \quad m = 1$

$y_h = c_1 e^{-3t} + c_2 e^{t}$

② guess y_p

$y_p = A\cos 3t + B\sin 3t$

③ Plug in

$y_p = A\cos 3t + B\sin 3t$

$y_p' = -3A\sin 3t + 3B\cos 3t$

$y_p'' = -9A\cos 3t - 9B\sin 3t$

$y_p'' + 2y_p' - 3y_p = 5\sin 3t$

$-9A\cos 3t - 9B\sin 3t + 2(-3A\sin 3t + 3B\cos 3t) - 3(A\cos 3t + B\sin 3t) = 5\sin 3t$

$(-9A + 6B - 3A)\cos 3t + (-9B - 6A - 3B)\sin 3t = 5\sin 3t$

$(-12A + 6B)\cos 3t + (-12B - 6A)\sin 3t = 5\sin 3t$

$\begin{cases} -12A + 6B = 0 \rightarrow 6B = 12A \rightarrow B = 2A \\ -6A - 12B = 5 \end{cases}$

$-6A - 24A = 5$

$-30A = 5$

$A = -1/6, \quad B = -1/3$

$\rightarrow y_p = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$

④ $\boxed{y = c_1 e^{-3t} + c_2 e^{t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$

2. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$$

eigenvalues: $\begin{vmatrix} 1-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 0(-2) = (1-\lambda)(3-\lambda) = 0$ $\boxed{\lambda_1 = 1, \ \lambda_2 = 3}$

$\lambda_1 = 1$: $\left(\begin{array}{cc|c} 1-1 & -2 & 0 \\ 0 & 3-1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 2 & 0 \end{array}\right)$ $R_1 + R_2 \rightarrow R_2$ $\left(\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow$ $-2x_2 = 0$, $x_2 = 0$, x_1 free $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\lambda_2 = 3$: $\left(\begin{array}{cc|c} 1-3 & -2 & 0 \\ 0 & 3-3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow$ $-2x_1 - 2x_2 = 0$, $2x_1 = -2x_2$, $x_1 = -x_2$ $\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

3. Consider the matrix

$$B = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

(a) Using the determinant formula

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix},$$

show that the characteristic polynomial of B is $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$.

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = -\lambda\begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= -\lambda\left((2-\lambda)(3-\lambda) - 0\cdot 1\right) - 0\left(\cancel{1\cdot(3-\lambda) - 1\cdot 1}^{\,0}\right) - 2\left(1\cdot 0 - 1(2-\lambda)\right)$$

$$= -\lambda(6 - 5\lambda + \lambda^2) + 2(2-\lambda)$$

$$= -6\lambda + 5\lambda^2 - \lambda^3 + 4 - 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \checkmark$$

(b) $p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$ can be factored as $p(\lambda) = -(\lambda-2)^2(\lambda-1)$. What are the eigenvalues of B?

$-(\lambda-2)^2(\lambda-1) = 0$

$\lambda_1 = 2 \qquad \lambda_2 = 1$

(c) Find the eigenvectors corresponding to the eigenvalues you found in part (b).

$$\left(\begin{array}{ccc|c} -\lambda & 0 & -2 & 0 \\ 1 & 2-\lambda & 1 & 0 \\ 1 & 0 & 3-\lambda & 0 \end{array}\right) \qquad \text{goal: } \left(\begin{array}{ccc|c} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\lambda_1 = 2$:

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right) \begin{array}{l} R_2 - R_1 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + x_3 = 0$
$x_1 = -x_3$
x_2 free

2 eigenvectors: $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

$\lambda_1 = 1$:

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{R_3 + R_1 \to R_1} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = -2x_3$
$x_2 = x_3$

$\boxed{\vec{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}$