Math 33B: Linear Systems

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Name: KEY

1. Solve the linear system $\mathbf{x}' = A\mathbf{x}$ where

(a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

① eigenvalues

$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i$

② eigenvectors

$\left(\begin{array}{cc|c} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \end{array}\right)$

$\lambda_1 = i$: $\left(\begin{array}{cc|c} -i & 1 & 0 \\ 1 & -i & 0 \end{array}\right) \to x_1 - ix_2 = 0$, $x_1 = ix_2$, $\vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\vec{v}_1 e^{\lambda_1 t} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} i \\ 1 \end{pmatrix} (\cos t + i\sin t)$

$= \begin{pmatrix} i(\cos t + i\sin t) \\ 1(\cos t + i\sin t) \end{pmatrix} = \begin{pmatrix} i\cos t - \sin t \\ \cos t + i\sin t \end{pmatrix}$

$= \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

③ $\boxed{\vec{x} = c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}$

(b) $A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$

① eigenvalues

$\begin{vmatrix} -\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda) - 6$

$= -\lambda + \lambda^2 - 6$

$= \lambda^2 - \lambda - 6$

$0 = (\lambda - 3)(\lambda + 2)$

$\lambda_1 = 3, \quad \lambda_2 = -2$

② eigenvectors: $\left(\begin{array}{cc|c} -\lambda & 2 & 0 \\ 3 & 1-\lambda & 0 \end{array}\right)$

$\lambda_1 = 3$ $\left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right)$

$3x_1 - 2x_2 = 0 \to x_1 = \frac{2}{3}x_2$

$\vec{v}_1 = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right)$ $2x_1 + 2x_2 = 0$, $x_1 = -x_2$ $\quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

③ $\boxed{\vec{x} = c_1\begin{pmatrix} 2/3 \\ 1 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}}$

(c) $A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$

① eigenvalues

$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) + 4$

$= 12 - 2\lambda - 6\lambda + \lambda^2 + 4$

$= \lambda^2 - 8\lambda + 16$

$0 = (\lambda - 4)^2$

$\lambda = 4$ mult. 2

② eigenvectors $\left(\begin{array}{cc|c} 2-\lambda & 4 & 0 \\ -1 & 6-\lambda & 0 \end{array}\right)$

$\left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & -2 & 0 \end{array}\right) \to -x_1 - 2x_2 = 0$, $x_1 = -2x_2$ $\quad \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\left(\begin{array}{cc|c} -2 & 4 & -2 \\ -1 & -2 & 1 \end{array}\right) \to -x_1 - 2x_2 = 1$

$x_1 + 1 = -2x_2$

$x_1 = -2x_2 - 1$ $\quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

③ $\boxed{\vec{x} = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{4t} + c_2\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) e^{4t}}$

2. Recall the equation of a mass-spring system:

$$mx''(t) + \mu x'(t) + kx(t) = F(t) \tag{1}$$

(a) For $F(t) = 0$, solve the system using second order methods. (*Note*: There are three types of solutions with different damping coefficients.)

$mx'' + \mu x' + k = 0$

characteristic: $m\lambda^2 + \mu\lambda + k = 0$

$\lambda = \dfrac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$

3 cases

(i) If $\mu^2 - 4\mu k > 0$, distinct, real roots "overdamped"

$$\boxed{x = c_1 e^{\left(\frac{-\mu + \sqrt{\mu^2 - 4km}}{2m}\right)t} + c_2 e^{\left(\frac{-\mu - \sqrt{\mu^2 - 4km}}{2m}\right)t}}$$

(ii) If $\mu^2 - 4km = 0$, real, repeated root, "critically damped"

$\lambda = \dfrac{-\mu}{2m} \rightarrow \boxed{x = c_1 e^{\left(\frac{-\mu}{2m}\right)t} + c_2 t e^{\left(\frac{-\mu}{2m}\right)t}}$

(iii) If $\mu^2 - 4mk < 0$, complex roots, "underdamped"

$\lambda = \dfrac{-\mu \pm \sqrt{-1(4mk - \mu^2)}}{2m} = \dfrac{-\mu \pm i\sqrt{4mk - \mu^2}}{2m} = \dfrac{-\mu}{2m} \pm i\dfrac{\sqrt{4mk - \mu^2}}{2m}$

$$\boxed{x = e^{\left(-\frac{\mu}{2m}\right)t}\left(c_1 \cos\left(\frac{\sqrt{4mk - \mu^2}}{2m}t\right) + c_2 \sin\left(\frac{\sqrt{4mk - \mu^2}}{2m}t\right)\right)}$$

(b) Using the substitution $x'(t) = v(t)$, we can rewrite (1) as a 2D linear system:

$$x'(t) = v(t)$$
$$v'(t) = -\frac{k}{m}x(t) - \frac{\mu}{m}v(t) + \frac{F(t)}{m},$$

which can then be written in matrix form

typo, should be

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{F(t)}{m} \end{pmatrix}$$

Find the solution to this system in the overdamped case , i.e. when $\mu^2 - 4km > 0$, when $F(t) = 0$. Is this the same solution as the overdamped case you found in (a)?

If $F(t) = 0$, the system is

$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$

① eigenvalues

$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} - \lambda \end{vmatrix} = -\lambda\left(-\frac{\mu}{m} - \lambda\right) + \frac{k}{m}$

$0 = \lambda^2 + \frac{\mu}{m}\lambda + \frac{k}{m}$

$0 = m\lambda^2 + \mu\lambda + k$ (Same characteristic equation as in part (a)!!!)

$\lambda = \dfrac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$ Given $\mu^2 - 4mk > 0$, real distinct roots

$\lambda_1 = \dfrac{-\mu + \sqrt{\mu^2 - 4mk}}{2m}$, $\lambda_2 = \dfrac{-\mu - \sqrt{\mu^2 - 4mk}}{2m}$

② eigenvectors $\left(\begin{array}{cc|c} -\lambda & 1 & 0 \\ -\frac{k}{m} & -\frac{\mu}{m} - \lambda & 0 \end{array}\right)$

$\lambda_1 = \dfrac{-\mu + \sqrt{\mu^2 - 4mk}}{2m}$, $\left(\begin{array}{cc|c} \frac{\mu - \sqrt{\mu^2 - 4mk}}{2m} & 1 & 0 \\ -\frac{k}{m} & -\frac{\mu}{m} + \frac{\mu - \sqrt{\mu^2 - 4mk}}{2m} & 0 \end{array}\right)$

First row says $x_2 = \left(\dfrac{-\mu + \sqrt{\mu^2 - 4mk}}{2m}\right)x_1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-\mu + \sqrt{\mu^2 - 4mk}}{2m} \end{pmatrix}$

$\lambda_2 = \dfrac{-\mu - \sqrt{\mu^2 - 4mk}}{2m}$ $\left(\begin{array}{cc|c} \frac{\mu + \sqrt{\mu^2 - 4mk}}{2m} & 1 & 0 \\ -\frac{k}{m} & -\frac{\mu}{m} + \frac{\mu + \sqrt{\mu^2 - 4mk}}{2m} & 0 \end{array}\right)$

First row says $x_2 = \left(\dfrac{-\mu - \sqrt{\mu^2 - 4mk}}{2m}\right)x_1$, $\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{-\mu - \sqrt{\mu^2 - 4km}}{2m} \end{pmatrix}$

③

So $\begin{pmatrix} x \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{-\mu + \sqrt{\mu^2 - 4mk}}{2m} \end{pmatrix} e^{\left(\frac{-\mu + \sqrt{\mu^2 - 4mk}}{2m}\right)t} + c_2 \begin{pmatrix} 1 \\ \frac{-\mu - \sqrt{\mu^2 - 4km}}{2m} \end{pmatrix} e^{\left(\frac{-\mu - \sqrt{\mu^2 - 4mk}}{2m}\right)t}$

First row: $x = c_1 e^{\left(\frac{-\mu + \sqrt{\mu^2 - 4mk}}{2m}\right)t} + c_2 e^{\left(\frac{-\mu - \sqrt{\mu^2 - 4km}}{2m}\right)t}$, same as (a)