Week 10 Solutions

Math 31AL

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Tuesday, December 3

1. $\int t^2(t^{1/2}-t^{-5})\,dt = \int (t^{5/2}-t^{-3})\,dt = \underbrace{\frac{t^{7/2}}{7/2} - \frac{t^{-2}}{-2} + C}_{\text{ok to stop here}} = \boxed{\frac{2}{7}t^{7/2} + \frac{2}{t^2} + C}$

distribute

2. (a) $\int_4^9 \frac{x-4}{\sqrt{x}+2}\,dx = \int_4^9 \frac{(\cancel{\sqrt{x}+2})(\sqrt{x}-2)}{\cancel{\sqrt{x}+2}}\,dx = \int_4^9 (x^{1/2}-2)\,dx = \left(\frac{x^{3/2}}{3/2} - 2x\right)\Big|_4^9 = \left(\frac{2}{3}x^{3/2} - 2x\right)\Big|_4^9$

* difference of squares $a^2-b^2=(a-b)(a+b)$

$= \overbrace{\frac{2}{3}9^{3/2} - 2(9) - \left(\frac{2}{3}\cdot 4^{3/2} - 2\cdot 4\right)}^{\text{on an exam stop here}} = \frac{2}{3}\cdot 9\sqrt{9} - 18 - \frac{2}{3}4\sqrt{4} + 8 = \frac{2}{3}\cdot 27 - 18 - \frac{2}{3}\cdot 8 + 8$

$9^{3/2} = 9\sqrt{9}$, $4^{3/2} = 4\sqrt{4}$

$= \frac{54}{3} - \frac{16}{3} - 10 = \frac{38}{3} - 10 = \frac{38}{3} - \frac{30}{3} = \frac{8}{3}$

(b) $\int_1^2 \frac{1-x}{x^4}\,dx = \int_1^2 \left(\frac{1}{x^4} - \frac{x}{x^4}\right)dx = \int_1^2 (x^{-4} - x^{-3})\,dx = \left(\frac{x^{-3}}{-3} - \frac{x^{-2}}{-2}\right)\Big|_1^2 = \left(\frac{-1}{3x^3} + \frac{1}{2x^2}\right)\Big|_1^2$

$= \frac{-1}{3\cdot 2^3} + \frac{1}{2\cdot 2^2} - \left(\frac{-1}{3\cdot 1^3} + \frac{1}{2\cdot 1^2}\right) = \underbrace{\frac{-1}{24} + \frac{1}{8} - \left(\frac{-1}{3} + \frac{1}{2}\right)}_{\text{on an exam stop here}} = \frac{-1}{24} + \frac{3}{24} + \frac{8}{24} - \frac{12}{24} = \frac{-2}{24} = \frac{-1}{12}$

(c) $\int_1^4 \frac{|x-3|}{x^3}\,dx = \int_1^3 \frac{|x-3|}{x^3}\,dx + \int_3^4 \frac{|x-3|}{x^3}\,dx = \int_1^3 \frac{-(x-3)}{x^3}\,dx + \int_3^4 \frac{x-3}{x^3}\,dx$

* $|x-3| = \begin{cases} x-3 & x \geq 3 \\ -(x-3) & x < 3 \end{cases}$

$= \int_1^3 \left(\frac{-1}{x^2} + \frac{3}{x^3}\right)dx + \int_3^4 \left(\frac{1}{x^2} - \frac{3}{x^3}\right)dx = \int_1^3 (-x^{-2} + 3x^{-3})\,dx + \int_3^4 (x^{-2} - 3x^{-3})\,dx$

$= \left(\frac{-x^{-1}}{-1} + \frac{3x^{-2}}{-2}\right)\Big|_1^3 + \left(\frac{x^{-1}}{-1} - \frac{3x^{-2}}{-2}\right)\Big|_3^4 = \left(\frac{1}{x} - \frac{3}{2x^2}\right)\Big|_1^3 + \left(\frac{-1}{x} + \frac{3}{2x^2}\right)\Big|_3^4$

$= \frac{1}{3} - \frac{3}{2\cdot 3^2} - \left(\frac{1}{1} - \frac{3}{2\cdot 1^2}\right) + \left(\frac{-1}{4} + \frac{3}{2\cdot 4^2}\right) - \left(\frac{-1}{3} + \frac{3}{2\cdot 3^2}\right)$ ← on an exam, stop here

$= \frac{1}{3} - \frac{1}{6} - 1 + \frac{3}{2} - \frac{1}{4} + \frac{3}{32} + \frac{1}{3} - \frac{1}{6}$

$= \frac{2}{3} - \frac{2}{6} - \frac{32}{32} + \frac{48}{32} - \frac{8}{32} + \frac{3}{32}$

$= \frac{1}{3} + \frac{11}{32} = \frac{32}{96} + \frac{33}{96} = \frac{65}{96}$

(3) Area $= \int_1^4 (x^2-5x+6)\,dx = \left(\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right)\Big|_1^4 = \frac{4^3}{3} - \frac{5\cdot 4^2}{2} + 6\cdot 4 - \left(\frac{1^3}{3} - \frac{5\cdot 1^2}{2} + 6\cdot 1\right)$ ← on an exam stop here

$= \frac{64}{3} - \frac{80}{2} + 24 - \frac{1}{3} + \frac{5}{2} - 6 = \frac{63}{3} - \frac{75}{2} + 18 = \frac{126}{6} - \frac{225}{6} + \frac{108}{6} = \frac{9}{6}$

$= \frac{3}{2}$ E

(4) I. $\frac{d}{dx}\left(\int_1^{\sqrt{1+x^2}} \frac{1}{t}\,dt\right) = \frac{1}{\sqrt{1+x^2}} \cdot \left(\sqrt{1+x^2}\right)' = \frac{1}{\sqrt{1+x^2}} \cdot \cancel{\frac{1}{2}}(1+x^2)^{-1/2}\cdot \cancel{2}x = \frac{x}{1+x^2}$ (C)

plug in $\sqrt{1+x^2}$

II. $\frac{d}{dx}\left(\int_0^{\sqrt{x}} \frac{dt}{1+t^2}\right) = \frac{1}{1+(\sqrt{x})^2} \cdot (\sqrt{x})' = \frac{1}{1+x} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{2(1+x)\sqrt{x}}$ (A)

plug in $\sqrt{x}$

III. $\frac{d}{dx}\left(\int_{\sqrt{x}}^{x} \frac{dt}{\sqrt{1+t^2}}\right) = \frac{d}{dx}\left(\int_0^x \frac{dt}{\sqrt{1+t^2}} + \int_{\sqrt{x}}^{0} \frac{dt}{\sqrt{1+t^2}}\right) = \frac{d}{dx}\left(\int_0^x \frac{dt}{\sqrt{1+t^2}} - \int_0^{\sqrt{x}} \frac{dt}{\sqrt{1+t^2}}\right)$

you can use any constant

$= \frac{1}{\sqrt{1+x^2}} \cdot (x)' - \frac{1}{\sqrt{1+\sqrt{x}^2}}(\sqrt{x})' = \frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x}} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{2\sqrt{1+x}} \cdot \frac{1}{\sqrt{x}}$

plug in x plug in $\sqrt{x}$

$= \frac{1}{\sqrt{1+x^2}} - \frac{1}{2\sqrt{x+x^2}}$ (B)

IV. $\frac{d}{dx}\left(\int_{x^2}^{x^4} \frac{dt}{\sqrt{1+t}}\right) = \frac{d}{dx}\left(\int_0^{x^4} \frac{dt}{\sqrt{1+t}} + \int_{x^2}^{0} \frac{dt}{\sqrt{1+t}}\right) = \frac{d}{dx}\left(\int_0^{x^4} \frac{dt}{\sqrt{1+t}} + \int_0^{x^2} \frac{dt}{\sqrt{1+t}}\right)$

you can use any constant

$= \frac{1}{\sqrt{1+x^4}} \cdot (x^4)' - \frac{1}{\sqrt{1+x^2}} \cdot (x^2)' = \frac{4x^3}{\sqrt{1+x^4}} - \frac{2x}{\sqrt{1+x^2}}$

plug in x^4 plug in x^2

(5) Water flows in: $1500 + 7t^2$ gal/hr

" " out: $4t + t^2$ gal/hr

Change in Water $= \int_1^5 (\text{rate in} - \text{rate out})\,dt$

$= \int_1^5 (1500 + 7t^2 - 4t - t^2)\,dt = \int_1^5 (1500 - 4t + 6t^2)\,dt$

$= \left(1500t - \frac{4t^2}{2} + \frac{6t^3}{3}\right)\Big|_1^5 = \left(1500t - 2t^2 + 2t^3\right)\Big|_1^5 = 1500(5) - 2(5)^2 + 2(5)^3 - \left(1500\cdot 1 - 2\cdot 1^2 + 2\cdot 1^3\right)$ ← on an exam stop here

$= 7500 - 50 + 250 - 1500 + \cancel{2} - \cancel{2} = 6000 + 200 = 6200$ gal

(6) $v(t) = (t-1)(t-2)(t-3) = (t^2-3t+2)(t-3) = t^3 - 3t^2 - 3t^2 + 9t + 2t - 6 = t^3 - 6t^2 + 11t - 6$

(a) displacement $= \int v(t)\,dt$

on an exam stop here

$s = \int_1^2 (t^3 - 6t^2 + 11t - 6)\,dt = \left(\frac{t^4}{4} - \cancel{\frac{6t^3}{3}}2t^3 + \frac{11t^2}{2} - 6t\right)\Big|_1^2 = \frac{16}{4} - 2(2)^3 + \frac{11\cdot 2^2}{2} - 6\cdot 2 - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right)$

$= 4 - 16 + 22 - 12 - \frac{1}{4} + 2 - \frac{11}{2} + 6 = 6 - \frac{1}{4} - \frac{11}{2} = \frac{24}{4} - \frac{1}{4} - \frac{22}{4} = \frac{1}{4}$

$s = \int_2^3 (t^3 - 6t^2 + 11t - 6)\,dt = \left(\frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t\right)\Big|_2^3 = \frac{3^4}{4} - 2\cdot 3^3 + \frac{11\cdot 3^2}{2} - 6\cdot 3 - \left(\frac{2^4}{4} - 2\cdot 2^3 + \frac{11\cdot 2^2}{2} - 6\cdot 2\right)$

$= \frac{81}{4} - 54 + \frac{99}{2} - 18 - \frac{16}{4} + 16 - 22 + 12 = \frac{81}{4} + \frac{198}{4} - \frac{16}{4} - 66 = \frac{263}{4} - \frac{264}{4} = \frac{-1}{4}$

$s = \int_1^3 (t^3 - 6t^2 + 11t - 6)\,dt = \int_1^2 (t^3 - 6t^2 + 11t - 6)\,dt + \int_2^3 (t^3 - 6t^2 + 11t - 6)\,dt$

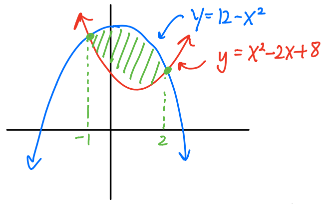
$= \frac{1}{4} - \frac{1}{4} = 0$

⑥ continued

(b) total distance $= \left|\int_1^2 (t^3-6t^2+11t-6)\,dt\right| + \left|\int_2^3 (t^3-6t^2+11t-6)\,dt\right| = \left|\frac{1}{4}\right| + \left|-\frac{1}{4}\right| = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

Thursday, December 5

①



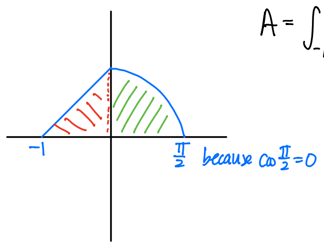
$$12-x^2 = x^2-2x+8$$
$$0 = 2x^2-2x-4$$
$$0 = 2(x^2-x-2)$$
$$0 = 2(x+1)(x-2)$$
$$x=-1 \qquad x=2$$

$$A = \int_{-1}^{2} \left((12-x^2)-(x^2-2x+8)\right)dx = \int_{-1}^{2}(12-x^2-x^2+2x-8)\,dx = \int_{-1}^{2}(-2x^2+2x+4)\,dx$$

$= \left(-\frac{2}{3}x^3+x^2+4x\right)\Big|_{-1}^{2} = -\frac{2}{3}(2)^3+2^2+4\cdot 2 - \left(-\frac{2}{3}(-1)^3+(-1)^2+4(-1)\right)$ ← on an exam stop here

$$= -\frac{16}{3}+4+8-\left(\frac{2}{3}+1-4\right) = -\frac{16}{3}+12-\frac{2}{3}+3 = -\frac{18}{3}+15 = -6+15 = 9$$

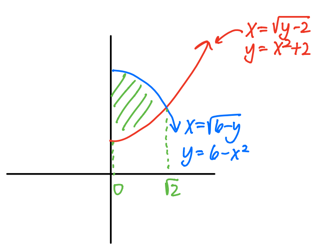
②



$$A = \int_{-1}^{0}(x+1)\,dx + \int_0^{\pi/2}\cos x\,dx = \left(\frac{x^2}{2}+x\right)\Big|_{-1}^{0} + \sin x\Big|_0^{\pi/2} = \frac{0^2}{2}+0-\left(\frac{(-1)^2}{2}+(-1)\right)+\sin\frac{\pi}{2}-\sin 0$$

$$= 0+0-\left(\frac{1}{2}-1\right)+1-0 = \frac{1}{2}+1 = \frac{3}{2}$$

③



Rewrite equations.

$x=\sqrt{y-2} \longrightarrow x^2=y-2 \longrightarrow y=x^2+2$

$x=\sqrt{6-y} \longrightarrow x^2=6-y \longrightarrow y=6-x^2$

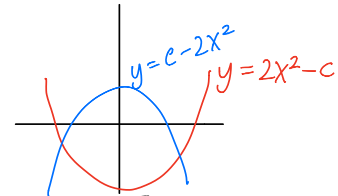
$$x^2+2 = 6-x^2$$
$$2x^2=4$$
$$x^2=2$$
$x=\pm\sqrt{2}$ want $x=\sqrt{2}$

$$A = \int_0^{\sqrt{2}}\left(6-x^2-(x^2+2)\right)dx = \int_0^{\sqrt{2}}(6-x^2-x^2-2)\,dx = \int_0^{\sqrt{2}}(4-2x^2)\,dx = 4x-\frac{2x^3}{3}\Big|_0^{\sqrt{2}}$$

$$= 4\sqrt{2}-\frac{2(\sqrt{2})^3}{3}-\left(4\cdot 0-\frac{2\cdot 0^3}{3}\right) = 4\sqrt{2}-\frac{2\cdot 2\sqrt{2}}{3}-0+0 = 4\sqrt{2}-\frac{4\sqrt{2}}{3} = \frac{12\sqrt{2}}{3}-\frac{4\sqrt{2}}{3} = \frac{8\sqrt{2}}{3}$$

④ Find where functions intersect:

$$c-2x^2 = 2x^2-c$$
$$2c = 4x^2$$
$$\frac{c}{2} = x^2$$
$$x = \pm\sqrt{\frac{c}{2}}$$



$$A = \int_{-\sqrt{c/2}}^{\sqrt{c/2}}\left(c-2x^2-(2x^2-c)\right)dx = \int_{-\sqrt{c/2}}^{\sqrt{c/2}}(2c-4x^2)\,dx = 2cx-\frac{4x^3}{3}\Big|_{-\sqrt{c/2}}^{\sqrt{c/2}} = 2c\sqrt{\frac{c}{2}}-\frac{4\left(\sqrt{\frac{c}{2}}\right)^3}{3}-\left(2c\left(-\sqrt{\frac{c}{2}}\right)-\frac{4\left(-\sqrt{\frac{c}{2}}\right)^3}{3}\right)$$

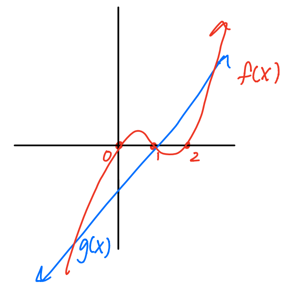
$$= 2c\sqrt{\frac{c}{2}}-\frac{4\cdot\frac{c}{2}\sqrt{\frac{c}{2}}}{3}-\left(-2c\sqrt{\frac{c}{2}}+\frac{4\frac{c}{2}\sqrt{\frac{c}{2}}}{3}\right) = 4c\sqrt{\frac{c}{2}}-\frac{\cancel{8}4\frac{c}{\cancel{2}}\sqrt{\frac{c}{2}}}{3} = \frac{12c\sqrt{\frac{c}{2}}}{3}-\frac{4c\sqrt{\frac{c}{2}}}{3} = \frac{8c\sqrt{\frac{c}{2}}}{3}$$

4 continued

$\frac{8c\sqrt{\frac{c}{2}}}{3} = \frac{16}{3} \longrightarrow 8c\sqrt{\frac{c}{2}} = 16 \longrightarrow c\sqrt{\frac{c}{2}} = 2 \longrightarrow c^2\cdot\frac{c}{2} = 4 \longrightarrow c^3 = 8 \longrightarrow c=2$

check: $\frac{8c\sqrt{\frac{c}{2}}}{3} = \frac{8\cdot 2\sqrt{\frac{2}{2}}}{3} = \frac{16\cdot\sqrt{1}}{3} = \frac{16}{3}$ ✓

5 $f(x) = x(x-1)(x-2) = x(x^2-3x+2) = x^3-3x^2+2x$
$g(x) = 3x-3$



$3x-3 = x(x-1)(x-2)$
$0 = x(x-1)(x-2) - 3x + 3$
$0 = x(x-1)(x-2) - 3(x-1)$
$0 = (x-1)[x(x-2)-3]$
$0 = (x-1)(x^2-2x-3)$
$0 = (x-1)(x-3)(x+1)$
intersect $x = 1, 3, -1$

$A = \int_{-1}^{1}[x(x-1)(x-2) - (3x-3)]\,dx + \int_1^3 (3x-3-x(x-1)(x-2))\,dx$

$= \int_{-1}^{1}(x^3-3x^2+2x-3x+3)\,dx + \int_1^3 (3x-3-(x^3-3x^2+2x))\,dx$

$= \int_{-1}^{1}(x^3-3x^2-x+3)\,dx + \int_1^3(-x^3+3x^2+x-3)\,dx$

$= \left(\frac{x^4}{4} - \frac{\cancel{3}x^3}{\cancel{3}} - \frac{x^2}{2} + 3x\right)\Big|_{-1}^{1} + \left(-\frac{x^4}{4} + \frac{\cancel{3}x^3}{\cancel{3}} + \frac{x^2}{2} - 3x\right)\Big|_1^3$

$= \frac{1}{4} - 1 - \frac{1}{2} + 3 - \left(\frac{1}{4} + 1 - \frac{1}{2} - 3\right) + \left(-\frac{81}{4} + 27 + \frac{9}{2} - 9\right) - \left(-\frac{1}{4} + 1 + \frac{1}{2} - 3\right)$ ← on an exam stop here

$= \cancel{\frac{1}{4}} - 1 - \cancel{\frac{1}{2}} + 3 - \cancel{\frac{1}{4}} - 1 + \cancel{\frac{1}{2}} + 3 - \frac{81}{4} + 27 + \frac{9}{2} - 9 + \frac{1}{4} - 1 - \frac{1}{2} + 3$

$= 24 - \frac{80}{4} + \frac{8}{2} = 24 - 20 + 4 = 8$

6 Average value of $f(x)$ on $[a,b]$ is $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$\frac{1}{4-1}\int_1^4 x^{1/2}\,dx = \frac{1}{3}\cdot\frac{x^{3/2}}{3/2}\Big|_1^4 = \frac{1}{3}\cdot\frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{9}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{9}\left(\sqrt{4^3} - 1\right) = \frac{2}{9}\left(\sqrt{64} - 1\right) = \frac{2}{9}(7) = \frac{14}{9}$

$f(c) = \sqrt{c} \longrightarrow \sqrt{c} = \frac{14}{9} \longrightarrow \boxed{c = \frac{196}{9}}$

7 $-x^2 + 9 = \frac{5}{2}x$
$0 = x^2 + \frac{5}{2}x - 9$

$0 = 2x^2 + 5x - 18$
$0 = 2x^2 + 9x - 4x - 18$
$= x(2x+9) - 2(2x+9)$
$= (x-2)(2x+9)$
$x = 2, -9/2$ want $x = 2$

$V = \int_0^2 \pi\left((-x^2+9)^2 - (\tfrac{5}{2}x)^2\right)dx = \pi\int_0^2\left(x^4 - 18x^2 + 81 - \tfrac{25}{4}x^2\right)dx$

$= \pi\int_0^2\left(x^4 - \frac{97}{4}x^2 + 81\right)dx = \pi\left(\frac{x^5}{5} - \frac{97}{12}x^3 + 81x\right)\Big|_0^2 = \pi\left(\frac{2^5}{5} - \frac{97}{\cancel{12}_{3}}\cdot\cancel{2^3}^{2} + 81\cdot 2 - 0\right)$

$= \pi\left(\frac{32}{5} - \frac{194}{3} + 162\right) = \pi\left(\frac{96}{15} - \frac{970}{15} + \frac{2430}{15}\right) = \frac{1556\pi}{15}$

⑧ $3\sin x = 1$

$\sin x = \frac{1}{3}$

$x = \sin^{-1}(\frac{1}{3}),\quad \pi - \sin^{-1}(\frac{1}{3})$

$\approx 0.34\ ,\quad 2.802$

$$V = \int_{\sin^{-1}(1/3)}^{\pi-\sin^{-1}(1/3)} \pi\left((3\sin x)^2 - 1^2\right)dx = \pi\int_{\sin^{-1}(1/3)}^{\pi-\sin^{-1}(1/3)} (9\sin^2 x - 1)\,dx$$

$* \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$V = \pi\int_{\sin^{-1}(1/3)}^{\pi-\sin^{-1}(1/3)}\left[\frac{9}{2}(1-\cos 2x) - 1\right]dx = \pi\int_{\sin^{-1}(1/3)}^{\pi-\sin^{-1}(1/3)}\left[\frac{7}{2} - \frac{9}{2}\cos 2x\right]dx$$

$$= \pi\left[\int_{\sin^{-1}(1/3)}^{\pi-\sin^{-1}(1/3)}\frac{7}{2}\,dx - \frac{9}{2}\underbrace{\int_{\sin^{-1}(1/3)}^{\pi-\sin^{-1}(1/3)}\cos 2x\,dx}_{u\text{-sub}}\right]$$

$$\int \cos 2x\,dx = \int \cos u\,\frac{du}{2} = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$$

$u = 2x$

$du = 2dx \rightarrow dx = \frac{du}{2}$

$$\text{so}\quad V = \pi\left(\frac{7}{2}x - \frac{9}{2}\cdot\frac{1}{2}\sin 2x\right)\Bigg|_{\sin^{-1}(1/3)}^{\pi-\sin^{-1}(1/3)} = \underbrace{\pi\left(\frac{7}{2}(\pi-\sin^{-1}(1/3)) - \frac{9}{4}\sin 2(\pi-\sin^{-1}(1/3)) - \left(\frac{7}{2}\sin^{-1}(\tfrac{1}{3}) - \frac{9}{4}\sin 2\sin^{-1}(1/3)\right)\right)}_{\text{on exam stop here}}$$