Week 2 Solutions

Math 33A
Written by Victoria Kala, vtkala@math.ucla.edu

① $$\left(\begin{array}{ccc|c} 3 & 6 & 14 & 22 \\ 7 & 14 & 30 & 46 \\ 4 & 8 & 7 & 6 \end{array}\right) \xrightarrow{R_1 - R_3 \to R_1} \left(\begin{array}{ccc|c} -1 & -2 & 7 & 16 \\ 7 & 14 & 30 & 46 \\ 4 & 8 & 7 & 6 \end{array}\right) \begin{array}{l} 7R_1 + R_2 \to R_2 \\ 4R_1 + R_3 \to R_3 \end{array} \left(\begin{array}{ccc|c} -1 & -2 & 7 & 16 \\ 0 & 0 & 79 & 158 \\ 0 & 0 & 35 & 70 \end{array}\right) \begin{array}{l} -R_1 \to R_1 \\ \frac{1}{79}R_2 \to R_2 \\ \frac{1}{35}R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -7 & -16 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} R_1 + 7R_2 \to R_1 \\ \\ R_3 - R_2 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\uparrow$ y is free

let $y = t \in \mathbb{R}$.

$x + 2y = -2 \to x = -2 - 2y = -2 - 2t$

$z = 2$

$$\boxed{\begin{pmatrix} -2-2t \\ t \\ 2 \end{pmatrix} \text{ where } t \in \mathbb{R}}$$

② Do smaller systems to see the pattern:

If $n=3$ $\quad \{ x_1 + x_3 = 2x_2 \longrightarrow \{ x_1 - 2x_2 + x_3 = 0 \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \end{array}\right)$

$\uparrow x_2 \ \uparrow x_3$ free

$x_2 = s,\ x_3 = t \to x_1 = 2x_2 - x_3 = 2s - t$

$$\begin{pmatrix} 2s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t$$

If $n=4$:
$$\begin{cases} x_1 + x_3 = 2x_2 \\ x_2 + x_4 = 2x_3 \end{cases} \longrightarrow \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + 2R_2 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right)$$

$\uparrow x_3 \ \uparrow x_4$ free

$x_3 = s,\ x_4 = t$ $\qquad x_1 = 3x_3 - 2x_4$
$\qquad x_2 = 2x_3 - x_4$

$$\begin{pmatrix} 3s-2t \\ 2s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} t$$

If $n=5$:
$$\begin{cases} x_1 + x_3 = 2x_2 \\ x_2 + x_4 = 2x_3 \\ x_3 + x_5 = 2x_4 \end{cases} \longrightarrow \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \\ x_3 - 2x_4 + x_5 = 0 \end{cases} \quad \left(\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right) \begin{array}{l} R_1 - R_3 \to R_1 \\ 2R_3 + R_2 \to R_2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right) \xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -4 & 3 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right)$$

$\uparrow x_4 \ \uparrow x_5$ free

$x_1 = 4x_4 - 3x_5$
$x_2 = 3x_3 - 2x_4$
$x_3 = 2x_3 - x_5$
$x_4 = s$
$x_5 = t$

$$\begin{pmatrix} 4s-3t \\ 3s-2t \\ 2s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} t$$

Based on this pattern, for arbitrary n:
$$\left(\begin{array}{cccccc|c} 1 & -2 & 1 & & & & 0 \\ & 1 & -2 & 1 & & & 0 \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & 1 & -2 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccccc|c} 1 & 0 & \cdots & -(n-1) & n-2 & & 0 \\ 0 & 0 & \cdots & -(n-2) & n-3 & & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 & 0 \end{array}\right)$$

Solution:
$$\begin{pmatrix} n-1 \\ n-2 \\ \vdots \\ 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -(n-2) \\ -(n-1) \\ \vdots \\ -1 \\ 0 \\ 1 \end{pmatrix} t$$

③ $a\begin{pmatrix}5\\4\\3\\-1\end{pmatrix}+b\begin{pmatrix}-2\\-2\\-1\\2\end{pmatrix}+c\begin{pmatrix}11\\6\\4\\1\end{pmatrix}=\begin{pmatrix}4\\3\\2\\-1\end{pmatrix}$

$$\left(\begin{array}{ccc|c}5&-2&11&4\\4&-2&6&3\\3&-1&4&2\\-1&2&1&-1\end{array}\right) \begin{array}{l}R_1\leftrightarrow R_4\end{array} \left(\begin{array}{ccc|c}-1&2&1&-1\\4&-2&6&3\\3&-1&4&2\\5&-2&11&4\end{array}\right) \begin{array}{l}4R_1+R_2\to R_2\\3R_1+R_3\to R_3\\5R_1+R_4\to R_4\end{array}$$

$$\left(\begin{array}{ccc|c}-1&2&1&-1\\0&6&10&-1\\0&5&7&-1\\0&8&16&-1\end{array}\right) \begin{array}{l}-R_1\to R_1\\R_2-R_3\to R_2\end{array} \left(\begin{array}{ccc|c}1&-2&-1&1\\0&1&3&0\\0&5&7&-1\\0&8&16&-1\end{array}\right) \begin{array}{l}-8\ \ -24\ \ 0\\ \ \ 8\ \ \ \ 16\ \ -1\\-5R_2+R_3\to R_3\\-8R_2+R_4\to R_4\end{array} \left(\begin{array}{ccc|c}1&-2&1&1\\0&1&3&0\\0&0&-8&-1\\0&0&-8&-1\end{array}\right) \begin{array}{l}R_4-R_3\to R_3\end{array}$$

$$\left(\begin{array}{ccc|c}1&-2&1&1\\0&1&3&0\\0&0&-8&-1\\0&0&0&0\end{array}\right) \begin{array}{l}R_3\to R_3\end{array} \left(\begin{array}{ccc|c}1&-2&1&1\\0&1&3&0\\0&0&1&1/8\\0&0&0&0\end{array}\right)$$

unique solution $\longrightarrow$ $\boxed{\text{yes, } \begin{pmatrix}4\\3\\2\\-1\end{pmatrix} \text{ is a linear combination of the given vectors}}$

④ (a) $\begin{pmatrix}2\cdot2+4\cdot3+1\cdot4+7\cdot5\\3\cdot2+2\cdot3-1\cdot4-1\cdot5\\4\cdot5-2\cdot3+5\cdot4+2\cdot5\end{pmatrix}=\begin{pmatrix}4+12+4+35\\6+6-4-5\\20-6+20+10\end{pmatrix}=\boxed{\begin{pmatrix}55\\3\\44\end{pmatrix}}$

(b) $\begin{pmatrix}1\cdot6+3\cdot2+5\cdot1\\-1\cdot6+2\cdot2-4\cdot1\end{pmatrix}+\begin{pmatrix}7\cdot7+3\cdot6+5\cdot3-1\cdot2\\4\cdot7-2\cdot6+1\cdot3+3\cdot2\end{pmatrix}=\begin{pmatrix}6+6+5\\-6+4-4\end{pmatrix}+\begin{pmatrix}49+18-15-2\\28-12+3+6\end{pmatrix}=\begin{pmatrix}17\\-6\end{pmatrix}+\begin{pmatrix}50\\25\end{pmatrix}$

$=\boxed{\begin{pmatrix}67\\19\end{pmatrix}}$

⑤ $\boxed{\text{Translations: not linear}}$

example: $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+1\\y\end{pmatrix}$

(shifts x component right 1 unit)

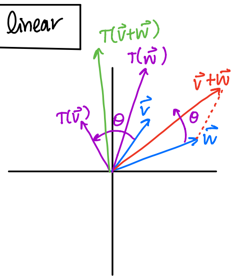
- $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow T\begin{pmatrix}1\\0\end{pmatrix}+T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$

  but $T\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)=T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$ not the same

- also $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$ and $5T\begin{pmatrix}1\\0\end{pmatrix}=5\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}10\\0\end{pmatrix}$

  but $T\left(5\begin{pmatrix}1\\0\end{pmatrix}\right)=T\begin{pmatrix}5\\0\end{pmatrix}=\begin{pmatrix}5+1\\0\end{pmatrix}=\begin{pmatrix}6\\0\end{pmatrix}$ not the same

$\boxed{\text{Rotations: linear}}$



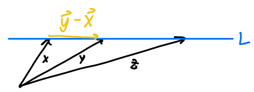
Idea (will prove in detail later in the course)
- If we rotate $\vec{v}$ and $\vec{w}$ counterclockwise by θ and add them together, will be the same as if we added $\vec{v}$ and $\vec{w}$ together first and then rotated counterclockwise by θ
- Similar for scaling, if we scale first then rotate it will be the same as if we rotated first then scaled

$\boxed{\text{Reflections: linear}}$

* as long as we reflect about a line passing through the origin

Idea (will prove in detail later in the course)
- If we reflect $\vec{v}$ and $\vec{w}$ then add, same as if we added $\vec{v}$ and $\vec{w}$ first then reflected
- If we reflect $\vec{v}$ and scale, same as if we scaled then reflected

⑥



equation of a line L: $\vec{r_0}+t\vec{v}$ where $\vec{r_0}$ is a "point" on the line
$\vec{v}$ is the direction

e.g. we could write the equation of this line as $\vec{x}+t(\vec{y}-\vec{x})$

Notice that at $t=0$ we have $\vec{x}+0(\vec{y}-\vec{x})=\vec{x}$

$t=1$ " " $\vec{x}+\vec{y}-\vec{x}=\vec{y}$

There is some t value so that $\vec{z}=\vec{x}+t(\vec{y}-\vec{x})$.

Q: if we take the transformation of a line, is it still a line?

$T(\text{line}) = T(\vec{x}+t(\vec{y}-\vec{x})) = T(\vec{x})+T(t(\vec{y}-\vec{x})) = \underbrace{T(\vec{x})}_{\text{"}\vec{r_0}\text{"}}+t\underbrace{T(\vec{y}-\vec{x})}_{\text{"}\vec{v}\text{"}}$ i.e. $\vec{r_0}+t\vec{v}$, still a line!

by property of linear transformation

Yes!