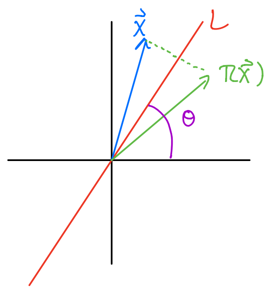
Week 3 Solutions

Math 33A

Written by Victoria Kala, vtkala@math.ucla.edu

① Reflection about a line L:



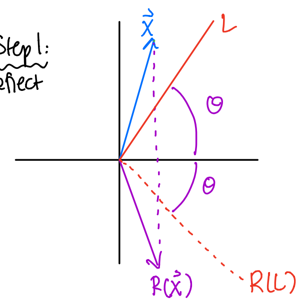
θ angle between L and x-axis

use dot product to find:

$\begin{pmatrix}1\\0\end{pmatrix}\cdot\vec{v} = \left\|\begin{pmatrix}1\\0\end{pmatrix}\right\| \|\vec{v}\| \cos\theta$ where $\vec{v}$ is a vector on the line L.

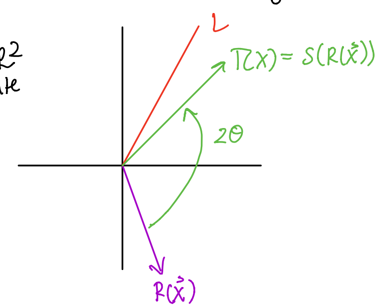
One idea: reflect $\vec{x}$ across the x-axis then rotate until we get $T(\vec{x})$

Step 1: reflect



$R(\vec{x}) = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\vec{x}$

Step 2: rotate



$S(\vec{x}) = \begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}$

* Note: you could have also defined a rotation first then a reflection

② $T_1(\vec{x}) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\vec{x}$, i.e. rotate counterclockwise by θ

$T_2(\vec{x}) = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\vec{x}$, i.e. rotate counterclockwise by φ

$$T_2(T_1(\vec{x})) = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\vec{x} = \begin{pmatrix}\cos\varphi\cos\theta - \sin\varphi\sin\theta & -\cos\varphi\sin\theta - \sin\varphi\cos\theta\\ \sin\varphi\cos\theta + \sin\theta\cos\varphi & -\sin\varphi\sin\theta + \sin\theta\cos\theta\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}\cos(\varphi+\theta) & -\sin(\varphi+\theta)\\ \sin(\varphi+\theta) & \cos(\varphi+\theta)\end{pmatrix}}_{\text{rotation } \checkmark}\vec{x}$$

③ $T_1(\vec{x}) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\vec{x}$, i.e. rotation

$T_2(\vec{x}) = \begin{pmatrix}a & b\\ b & -a\end{pmatrix}\vec{x}$, i.e. reflection (see textbook pg 70 section 2.3 for general formulas)

$$T_2(T_1(\vec{x})) = \begin{pmatrix}a & b\\ b & -a\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\vec{x} = \begin{pmatrix}a\cos\theta + b\sin\theta & -a\sin\theta + b\cos\theta\\ b\cos\theta - a\sin\theta & -b\sin\theta - a\cos\theta\end{pmatrix}\vec{x}$$

$$T_1(T_2(\vec{x})) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}a & b\\ b & -a\end{pmatrix}\vec{x} = \begin{pmatrix}a\cos\theta - b\sin\theta & b\cos\theta + a\sin\theta\\ a\sin\theta + b\cos\theta & b\sin\theta - a\cos\theta\end{pmatrix}\vec{x}$$

not the same

$T_3(\vec{x}) = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\vec{x}$, another rotation

$$T_3(T_1(\vec{x})) = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\vec{x} = \begin{pmatrix}\cos\varphi\cos\theta - \sin\varphi\sin\theta & -\cos\varphi\sin\theta - \sin\varphi\cos\theta\\ \sin\varphi\cos\theta + \sin\theta\cos\theta & -\sin\varphi\sin\theta + \cos\varphi\cos\theta\end{pmatrix}\vec{x}$$

$$T_1(T_3(\vec{x})) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\vec{x} = \begin{pmatrix}\cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi\\ \sin\theta\cos\varphi - \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi\end{pmatrix}\vec{x}$$

same

④ (a) $\begin{pmatrix} 2 & 4 & 1 & 7 \\ 3 & 2 & -1 & -1 \\ 4 & -2 & 5 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 1 & -7 \\ -2 & 9 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2(3)+4(1)+1(-2)+7(2) & 2(4)+4(-7)+1(9)+7(0) \\ 3(3)+2(1)-1(-2)-1(2) & 3(4)+2(-7)-1(9)-1(0) \\ 4(3)-2(1)+5(-2)+2(2) & 4(4)-2(-7)+5(9)+2(0) \end{pmatrix}$

$= \begin{pmatrix} 6+4-2+14 & 8-28+9+0 \\ 9+2+2-2 & 12-14-9-0 \\ 12-2-10+4 & 16+14+45+0 \end{pmatrix} = \begin{pmatrix} 22 & -11 \\ 11 & -11 \\ 4 & 75 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & 5 \\ -1 & 2 & -4 \end{pmatrix}\underbrace{\begin{pmatrix} 6 & 7 & 8 \\ 2 & 1 & 0 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 5 & -2 \\ 2 & 6 \end{pmatrix}} = \begin{pmatrix} 1 & 3 & 5 \\ -1 & 2 & -4 \end{pmatrix}\begin{pmatrix} 6(1)+7(5)+8(2) & 6(3)+7(-2)+8(6) \\ 2(1)+1(5)+0(2) & 2(3)+1(-2)+0(6) \\ 1(1)-1(5)-3(2) & 1(1)-1(-2)-3(6) \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 & 5 \\ -1 & 2 & -4 \end{pmatrix}\begin{pmatrix} 6+35+16 & 18-14+48 \\ 2+5+0 & 6-2+0 \\ 1-5-6 & 1+2-18 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ -1 & 2 & -4 \end{pmatrix}\begin{pmatrix} 57 & 52 \\ 7 & 4 \\ -10 & -15 \end{pmatrix} = \begin{pmatrix} 1(57)+3(7)+5(-10) & 1(52)+3(4)+5(-15) \\ -1(57)+2(7)-4(-10) & -1(52)+2(8)-4(-15) \end{pmatrix}$

$= \begin{pmatrix} 28 & -11 \\ -3 & 24 \end{pmatrix}$

⑤ (a) $A^2 = \begin{pmatrix} 1/2 & -\sqrt{3}/3 \\ \sqrt{3}/3 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & -\sqrt{3}/3 \\ \sqrt{3}/3 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\frac{1}{2}) - \frac{\sqrt{3}}{3}(\frac{\sqrt{3}}{3}) & \frac{1}{2}(-\frac{\sqrt{3}}{3}) - \frac{\sqrt{3}}{3}(\frac{1}{2}) \\ \frac{\sqrt{3}}{3}(\frac{1}{2}) + \frac{1}{2}(\frac{\sqrt{3}}{3}) & \frac{\sqrt{3}}{3}(-\frac{\sqrt{3}}{3}) + \frac{1}{2}(\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} -1/12 & -\sqrt{3}/3 \\ \sqrt{3}/3 & -1/12 \end{pmatrix}$

(b) * I am not sure what was intended for this question so take my solution w/ a grain of salt

rephrasing the question: is there a $\underbrace{\text{standard transformation}}_{\text{represented by } B}$ that performed twice is the same as another $\underbrace{\text{standard transformation}}_{\text{represented by } A}$?

A is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, looks like rotation and scaling

scaling factor: $\sqrt{a^2+b^2} = \sqrt{\frac{1}{4}+\frac{1}{3}} = \sqrt{\frac{1}{7}}$

$A = \frac{1}{\sqrt{7}}\begin{pmatrix} \sqrt{7}\cdot\frac{1}{2} & \sqrt{7}\cdot-\frac{\sqrt{3}}{3} \\ \sqrt{7}\cdot\frac{\sqrt{3}}{3} & \sqrt{7}\cdot\frac{1}{2} \end{pmatrix}$ $\cos\theta = \frac{\sqrt{7}}{2}$ impossible b.c. $\frac{\sqrt{7}}{2} > 1$ $\Rightarrow$ A is <u>not</u> a rotation and scaling.

A not standard transformation, don't think we can find standard transformation B s.t. $B^2 = A$.

* If $A = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ then $A = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$ i.e. rotation c.c. by $\frac{\pi}{3}$

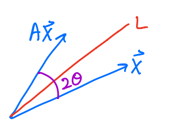
take $B = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}$, i.e. rotation c.c. by $\frac{\pi}{6}$

(c) $A^2 = \begin{pmatrix} 1/2 & \sqrt{3}/3 \\ \sqrt{3}/3 & -1/2 \end{pmatrix}\begin{pmatrix} 1/2 & \sqrt{3}/3 \\ \sqrt{3}/3 & -1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{3} & \frac{\sqrt{3}}{6}-\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6}-\frac{\sqrt{3}}{6} & \frac{1}{3}+\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{7}{12} & 0 \\ 0 & \frac{7}{12} \end{pmatrix}$

A looks like reflection about a line, $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, but $a^2+b^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 \neq 1$ $\therefore$ not standard transformation

don't think we can find standard transformation B s.t. $B^2 = A$.

* If $A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, then A would be a reflection about a line L



one option would be to take $B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, i.e. rotation c.c. by θ

where 2θ is the angle between $\vec{x}$ and $A\vec{x}$