Winter 2020 Math 33A
Week 3 Worksheet
Harris Khan

Problem 1: Find all 2 by 2 matrices X with

$$\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix} X = \begin{bmatrix}3 & 5\\ 5 & 8\end{bmatrix}$$

Hint: Begin by writing $X = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Translate the above equation into a system of 4 equations in the variables a, b, c, d.

$$\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}3 & 5\\ 5 & 8\end{pmatrix}$$

$$\begin{pmatrix}a+2c & b+2d\\ 2a+3c & 2b+3d\end{pmatrix} = \begin{pmatrix}3 & 5\\ 5 & 8\end{pmatrix}$$

$a+2c=3$
$b+2d=5$
$2a+3c=5$
$2b+3d=8$

$$\left(\begin{array}{cccc|c}1&0&2&0&3\\0&1&0&2&5\\2&0&3&0&5\\0&2&0&3&8\end{array}\right)\begin{array}{l}\\ \\ -2R_1+R_3\to R_3\\ -2R_2+R_4\to R_4\end{array}\left(\begin{array}{cccc|c}1&0&2&0&3\\0&1&0&2&5\\0&0&-1&0&-1\\0&0&0&-1&-2\end{array}\right)\begin{array}{l}2R_3+R_1\to R_1\\ 2R_4+R_2\to R_2\\ \\ \\ \end{array}\left(\begin{array}{cccc|c}1&0&0&0&1\\0&1&0&0&1\\0&0&-1&0&-1\\0&0&0&-1&-2\end{array}\right)\begin{array}{l}\\ \\ -R_3\to R_3\\ -R_4\to R_4\end{array}$$

$$\left(\begin{array}{cccc|c}1&0&0&0&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&2\end{array}\right)$$

$a=1$
$b=1$
$c=1$
$d=2$

$$\boxed{X=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}}$$

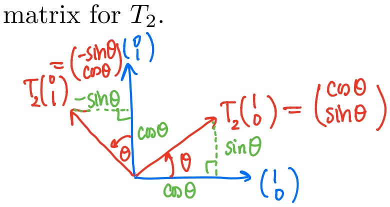
Check: $\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} = \begin{pmatrix}1+2 & 1+4\\ 2+3 & 2+6\end{pmatrix} = \begin{pmatrix}3 & 5\\ 5 & 8\end{pmatrix}$ ✓

Problem 2: Recall that to find the matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. a 2 by 2 matrix A with $T(x) = Ax$, we can get the first column by simply computing $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$, and we can get the second column by computing $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

a) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of scaling by a factor of 2. Find the matrix for T_1.

$T_1\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1\\2x_2\end{pmatrix}$ $\quad T_1\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$ $\quad T_1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} \longrightarrow \boxed{A = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}}$ $\quad$ i.e. $T_1(\vec{x}) = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\vec{x}$

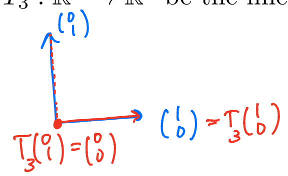
b) Let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of rotation by an angle of θ counterclockwise. Find the matrix for T_2.



$$\boxed{A = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}}$$

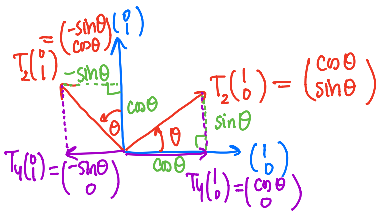
i.e. $T_2(\vec{x}) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\vec{x}$

c) Let $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of projection onto the x-axis. Find the matrix of T_3.



$\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$ i.e. $T_3(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\vec{x}$

d) Let $T_4 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of rotating by an angle of θ counterclockwise, and then projecting onto the x-axis. Find the matrix of T_4.



$\boxed{A = \begin{pmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{pmatrix}}$ i.e. $T_4(\vec{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{pmatrix}\vec{x}$

e) Verify that if you multiply your matrix from c with the matrix from $\cancel{d}$ b, in that order, you get the matrix from d. Why is this true? And why is the matrix multiplication done in that order?

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{project second}}\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotate first}} = \begin{pmatrix} \cos\theta + 0 & -\sin\theta + 0 \\ 0+0 & 0+0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{pmatrix}$$ same as (d) ✓

If we did $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then this means project first, rotate second which is not the same.

f) Which of the linear transformations from a through d are invertible? *Hint*: Can the transformation be undone?

(a) and (b) are invertible because they can be undone

(c) and (d) are not invertible because they cannot be undone

Problem 3: In this problem, we show 2 by 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are invertible (i.e. have rank 2) if and only if $ad - bc \neq 0$. (This quantity is known as the *determinant* of the matrix. We will eventually see a notion of determinant for larger matrices as well, and it will similarly tell us about invertibility.)

a) If $a = 0$ and $c = 0$, is the matrix invertible?

A matrix is invertible if $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|c} 0 & b & 0 \\ 0 & d & 0 \end{array}\right)$$

x_1 is a free variable $\Rightarrow$ infinitely many solutions

$\Rightarrow$ $\vec{x} = \vec{0}$ is not the only solution

$\Rightarrow$ $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ not invertible

b) Suppose $a \neq 0$. Then we can pick it as a pivot, divide through by a and get the matrix

$$\begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix}$$

Perform the next step in row reduction. Can we figure out when the matrix is invertible without going further? (Note that we're ready to pick another pivot, if we can).

Try looking at $A\vec{x} = \vec{0}$:

$$\left(\begin{array}{cc|c} 1 & b/a & 0 \\ c & d & 0 \end{array}\right) \xrightarrow{-cR_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & b/a & 0 \\ 0 & -\frac{bc}{a} + d & 0 \end{array}\right)$$ want only one solution

need $-\frac{bc}{a} + d \neq 0 \Rightarrow ad \neq bc$

c) Suppose $a = 0$ but $c \neq 0$. Then we can just swap rows and do the same process as part *b*. Verify this gives you the same condition as part *b*.

$$\left(\begin{array}{cc|c} 0 & b & 0 \\ c & d & 0 \end{array}\right) \xrightarrow{\frac{1}{c}R_2 \to R_2} \left(\begin{array}{cc|c} 0 & b & 0 \\ 1 & d/c & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & d/c & 0 \\ 0 & b & 0 \end{array}\right)$$

want only one solution

need $b \neq 0$

$\Updownarrow$

$bc \neq 0$

$\Updownarrow$

$bc \neq 0 \cdot d$

$\Updownarrow$

$bc \neq ad$ since $a = 0$.

d) Which of the matrices from 2*a* through 2*d* are invertible? Does this agree with your answer to 2*f*?

2a: $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\det A = 2(2) - 0(0) = 4 \neq 0$ so invertible

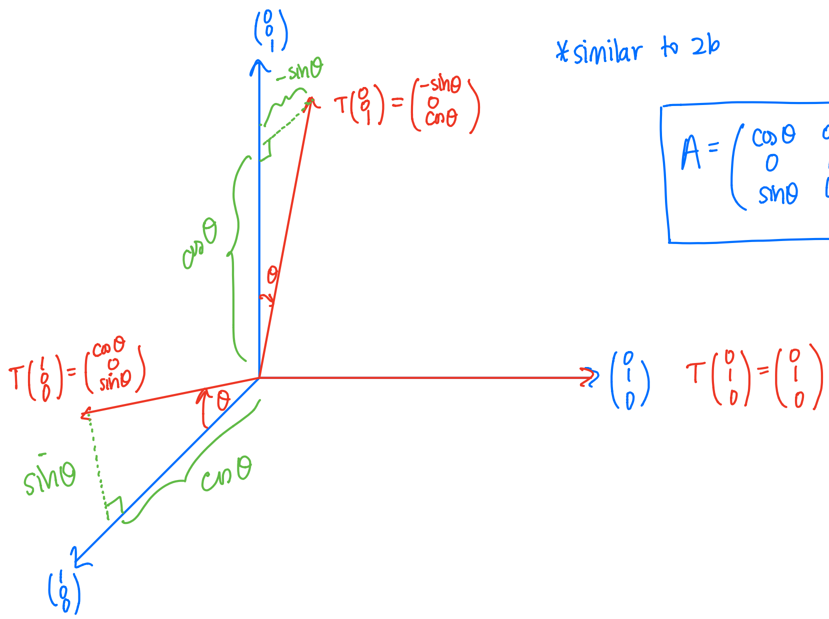
2b: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\det A = \cos\theta\cos\theta - \sin\theta(-\sin\theta) = \cos^2\theta + \sin^2\theta = 1 \neq 0$ so invertible

2c: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\det A = 1 \cdot 0 - 0 \cdot 0 = 0$, not invertible

2d: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{pmatrix}$ $\det A = \cos\theta \cdot 0 - \sin\theta \cdot 0 = 0$, not invertible

Problem 4: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation by an angle of θ on the xz-plane (the rotation is going from the positive x-axis towards the positive z-axis). Find the matrix of T.

Hint: Finding $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$, and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ will give us the first, second and third columns of this matrix. To figure out what will happen to the first and third, draw the xz-plane and what happens to these vectors after the rotation occurs.



*similar to 2b

$$\boxed{A = \begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}}$$