Week 4 Solutions

Math 33A

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① (a) $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\boxed{A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}}$ Check: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

* can also use the 2x2 formula for (a)

(b) $\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\boxed{A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

(c) $\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{2R_2 + R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right) \xrightarrow{R_2 - R_3 \to R_3}$

$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right) \xrightarrow{-R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right)$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}$ $\boxed{A^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 1 \\ -2 & 2 & 1 \end{pmatrix}}$

② $T(\vec{x}) = A\vec{x}$ $T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $T\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

i.e. $T\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow A\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{4-6}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

$\boxed{T(\vec{x}) = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\vec{x}}$

③ (a) True, need rank $A = n$ to be invertible

(b) False, A needs to be square (i.e. $n \times n$ or $m \times m$)

(c) True e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\det A = 4 - 6 = -2 \neq 0$, A invertible; $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $\det A^T = 4 - 6 = -2 \neq 0$, A^T invertible

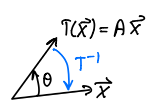
* note: examples are not proofs but can be useful

④ (a) $\begin{vmatrix} 1 & 5 \\ 4 & 2 \end{vmatrix} = 1 \cdot 2 - 4 \cdot 5 = 2 - 20 = -18 \neq 0$ <u>invertible</u>

(b) $\begin{vmatrix} 1/2 & 2 \\ 1 & 4 \end{vmatrix} = \frac{1}{2} \cdot 4 - 1 \cdot 2 = 2 - 2 = 0$ <u>not invertible</u>

(c) $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \cdot 0 - 0 \cdot 1 = 0 - 0 = 0$ <u>not invertible</u>

⑤ $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



geometrically, we expect T^{-1} to be clockwise rotation,

i.e. $T(\vec{x}) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}\vec{x} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\vec{x}$

since $\cos\theta = \cos(-\theta)$, $-\sin\theta = \sin(-\theta)$

using formula: $A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

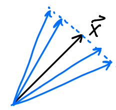
✓ same

⑥ General 2x2 projection matrix: $A=\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}$ where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a unit vector parallel to some line.

$$\det A = u_1^2 u_2^2 - u_1u_2\, u_1u_2 = 0 \text{, not invertible}$$

* projection matrices are not invertible!

Geometrically:



Given any vector $\vec{x}$, can I undo projection?
No, there are infinitely many vectors whose projections are all $\vec{x}$!