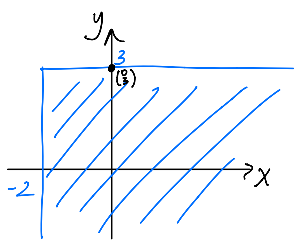
Week 6 Solutions

Math 33A
Written by Victoria Kala, vtkala@math.ucla.edu

①



Need to show U is either not closed under addition or scalar multiplication.

Not closed under scalar multiplication: $\begin{pmatrix}0\\3\end{pmatrix} \in U$, but $2\begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}0\\6\end{pmatrix} \notin U$.

$\Rightarrow$ not a subspace

② (a) $\left(\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 1 & 5 & -9 & 0 \\ 4 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 5 & -9 & 0 \\ 3 & 2 & -1 & 0 \\ 4 & 1 & 2 & 0 \end{array}\right) \begin{matrix} -3R_1+R_2 \to R_2 \\ -4R_1+R_3 \to R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 5 & -9 & 0 \\ 0 & -13 & 26 & 0 \\ 0 & -19 & 38 & 0 \end{array}\right) \begin{matrix} -\frac{1}{13}R_2 \to R_2 \\ -\frac{1}{19}R_3 \to R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 5 & -9 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) R_3 - R_2 \to R_2$

$\left(\begin{array}{ccc|c} 1 & 5 & -9 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-5R_2+R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -19 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ let $x_3 = t$. $\begin{matrix} x_1 = 19x_3 \\ x_2 = -2x_3 \end{matrix}$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19 \\ -2 \\ 1 \end{pmatrix} x_3$

free variable

1st and 2nd columns form basis of $\operatorname{im} T$

$\boxed{\text{basis of } \operatorname{im} T = \left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\} \qquad \text{basis of } \ker T = \left\{ \begin{pmatrix} 19 \\ -2 \\ 1 \end{pmatrix} \right\}}$

$\operatorname{im} T = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\} \qquad \ker T = \operatorname{span}\left\{ \begin{pmatrix} 19 \\ -2 \\ 1 \end{pmatrix} \right\}$

(b) No. The vectors in $\operatorname{im} T$ are in $\mathbb{R}^3$, so $\operatorname{im} T \neq \mathbb{R}^2$. $\dim(\operatorname{im} T) = 2$ means $\operatorname{im} T$ is a plane in $\mathbb{R}^3$

③ $T: \mathbb{R}^5 \to \mathbb{R}^3$ since A is 3×5 matrix. Vectors in $\ker T$ are in $\mathbb{R}^5$, so $\ker T \neq \mathbb{R}^2$. But we can say that $\dim(\ker T) = 5 - \dim(\operatorname{im} T) = 5 - 3 = 2$ by the rank-nullity theorem, so $\ker T$ is a two-dimensional plane in $\mathbb{R}^5$.

④ (a) S: $c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, check if $c_1 = c_2 = 0$.

$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right) \begin{matrix} -2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \end{matrix} \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & -3 & 0 \\ 0 & -6 & 0 \end{array}\right) \begin{matrix} -\frac{1}{3}R_2 \to R_2 \\ -\frac{1}{6}R_3 \to R_3 \end{matrix} \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \begin{matrix} R_1 - 4R_2 \to R_2 \\ R_3 - R_2 \to R_3 \end{matrix} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $c_1 = c_2 = 0$

$\boxed{S \text{ is linearly independent}}$

T: $c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ check if $c_1 = c_2 = c_3 = 0$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) R_3 - R_2 \to R_3 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $c_1 = c_2 = c_3 = 0$

$\boxed{T \text{ is linearly independent}}$

U: $c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ check if $c_1 = c_2 = c_3 = 0$

$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \begin{matrix} R_1 - R_2 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ This system has infinitely many solutions, so $c_1 = c_2 = c_3 = 0$ is not the only sol.

$\Rightarrow$ $\boxed{\begin{matrix} U \text{ is } \underline{\text{not}} \text{ linearly independent} \\ \text{(linearly dependent)} \end{matrix}}$

(b) Check if pivot in every row.

S: in (a) after reducing we had $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ only 2 pivots $\Rightarrow$ $\boxed{S \text{ is not a basis of } \mathbb{R}^3}$

T: in (a) after reducing we had $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \end{array}\right)$ pivot in every row ✓ $\Rightarrow$ $\boxed{\text{T is a basis of } \mathbb{R}^3}$

U: not linearly independent $\Rightarrow$ $\boxed{\text{U is not a basis of } \mathbb{R}^3}$

⑤ Model w/ vectors

(a) $1+2t \to \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\quad 4+5t+t^2 \to \begin{pmatrix}4\\5\\1\end{pmatrix}$ $\quad 1-t+7t^2 \to \begin{pmatrix}1\\-1\\7\end{pmatrix}$

$c_1\begin{pmatrix}1\\2\\0\end{pmatrix}+c_2\begin{pmatrix}4\\5\\1\end{pmatrix}+c_3\begin{pmatrix}1\\-1\\7\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ check if $c_1=c_2=c_3=0$

$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 2 & 5 & -1 & 0 \\ 0 & 1 & 7 & 0 \end{array}\right)_{-2R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 1 & 7 & 0 \end{array}\right)_{-\frac{1}{3}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 7 & 0 \end{array}\right)_{R_3-R_2\to R_3} \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ has solution $c_1=c_2=c_3=0$

linearly independent

(b) $-1+t \to \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ $\quad 2+t^2 \to \begin{pmatrix}2\\0\\1\\0\end{pmatrix}$ $\quad -3+t+t^2 \to \begin{pmatrix}-3\\1\\1\\0\end{pmatrix}$ $\quad -1-t-t^3 \to \begin{pmatrix}-1\\-1\\0\\-1\end{pmatrix}$ $\quad 2+3t^2+t^3 \to \begin{pmatrix}2\\0\\3\\1\end{pmatrix}$

$c_1\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}2\\0\\1\\0\end{pmatrix}+c_3\begin{pmatrix}-3\\1\\1\\0\end{pmatrix}+c_4\begin{pmatrix}-1\\-1\\0\\-1\end{pmatrix}+c_5\begin{pmatrix}2\\0\\3\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ check if $c_1=c_2=c_3=c_4=c_5=0$

$\left(\begin{array}{ccccc|c} -1 & 2 & -3 & -1 & 2 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right)$ This system has more columns than rows $\Rightarrow$ infinitely many solutions, so $c_1=\dots=c_5=0$ not only sol.

linearly dependent

(c) $\begin{pmatrix}1&2\\0&1\end{pmatrix}\to\begin{pmatrix}1\\2\\0\\1\end{pmatrix}$, $\begin{pmatrix}2&0\\1&3\end{pmatrix}\to\begin{pmatrix}2\\0\\1\\3\end{pmatrix}$, $\begin{pmatrix}1&1\\4&0\end{pmatrix}\to\begin{pmatrix}1\\1\\4\\0\end{pmatrix}$ (this is just one way)

$c_1\begin{pmatrix}1\\2\\0\\1\end{pmatrix}+c_2\begin{pmatrix}2\\0\\1\\3\end{pmatrix}+c_3\begin{pmatrix}1\\1\\4\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ check if $c_1=c_2=c_3=0$

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right)_{\substack{R_2-2R_1\to R_2 \\ R_4-R_1\to R_4}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)_{R_2\leftrightarrow R_4} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -4 & -1 & 0 \end{array}\right)_{\substack{R_3-R_2\to R_3 \\ 4R_2+R_4\to R_4}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right)_{\substack{\frac{1}{5}R_3\to R_3 \\ \frac{1}{3}R_4\to R_4}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \cdots$

this system has solution $c_1=c_2=c_3=0$ $\Rightarrow$ linearly independent

⑥ $\{\vec{u},\vec{v}\}$ linearly independent means $c_1\vec{u}+c_2\vec{v}=\vec{0} \Rightarrow c_1=c_2=0$

(a) $\{\vec{u},\vec{v},\vec{w}\}$ linearly dependent means $c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=\vec{0}$ where c_1,c_2,c_3 not all 0.

Solve for $\vec{w}$: $c_3\vec{w}=-c_1\vec{u}-c_2\vec{v}$ need to make sure $c_3\neq 0$. If $c_3=0$ then $c_1\vec{u}+c_2\vec{v}=0\Rightarrow c_1=c_2=0$, but

$\vec{w}=\frac{-c_1}{c_3}\vec{u}-\frac{c_2}{c_3}\vec{v}$ $\quad c_1,c_2,c_3$ cannot all be 0. So $c_3\neq 0$

$\vec{w}\in\text{span}\{\vec{u},\vec{v}\}$ ✓ yes

(b) $\vec{w}\in\text{span}\{\vec{u},\vec{v}\}$ means $\vec{w}=a\vec{u}+b\vec{v}$ for some $a,b\in\mathbb{R}$.

$a\vec{u}+b\vec{v}-\vec{w}=\vec{0}$

$c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=\vec{0}$ where $c_1=a$, $c_2=b$, $c_3=-1$. $\Rightarrow$ linearly dependent

⑦ (a) $c_1\begin{pmatrix}1\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right)$ $\boxed{c_1=5,\ c_2=-2}$

(b) $[\vec{x}]_{\mathcal{B}}=\begin{pmatrix}5\\-2\end{pmatrix}$

(c) $c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix}\to\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 2 & -1 & -2 \end{array}\right)_{-2R_1+R_2\to R_2}\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & -1 & -12 \end{array}\right)_{-R_2\to R_2}\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 12 \end{array}\right)$ $\quad c_1=5$, $c_2=12$ $\quad \boxed{[\vec{x}]_{\mathcal{C}}=\begin{pmatrix}5\\12\end{pmatrix}}$