Week 7 Solutions

Math 33A
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① $\vec{v_1} = \begin{pmatrix}2\\2\\1\end{pmatrix}$, $\vec{v_2} = \begin{pmatrix}3\\3\\6\end{pmatrix}$

$\vec{w_1} = \vec{v_1} = \begin{pmatrix}2\\2\\1\end{pmatrix}$

$$\vec{w_2} = \vec{v_2} - \text{proj}_{\vec{w_1}}(\vec{v_2}) = \begin{pmatrix}3\\3\\6\end{pmatrix} - \frac{\begin{pmatrix}3\\3\\6\end{pmatrix}\cdot\begin{pmatrix}2\\2\\1\end{pmatrix}}{\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\2\\1\end{pmatrix}}\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}3\\3\\6\end{pmatrix} - \frac{18}{9}\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}3\\3\\6\end{pmatrix} - 2\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\4\end{pmatrix}$$

Check $\vec{w_1}\cdot\vec{w_2} = \begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\\4\end{pmatrix} = -2-2+4 = 0$ ✓

$$\vec{u_1} = \frac{1}{\sqrt{2^2+2^2+1^2}}\begin{pmatrix}2\\2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix} = \boxed{\begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix}} = \vec{u_1}$$

$$\vec{u_2} = \frac{1}{\sqrt{1^2+1^2+4^2}}\begin{pmatrix}-1\\-1\\4\end{pmatrix} = \frac{1}{\sqrt{18}}\begin{pmatrix}-1\\-1\\4\end{pmatrix} = \boxed{\begin{pmatrix}-1/\sqrt{18}\\-1/\sqrt{18}\\4/\sqrt{18}\end{pmatrix}} = \vec{u_2}$$

② $\vec{v_1} = \begin{pmatrix}4\\4\\2\end{pmatrix}$, $\vec{v_2} = \begin{pmatrix}-4\\2\\4\end{pmatrix}$, $\vec{v_3} = \begin{pmatrix}36\\0\\0\end{pmatrix}$

$\vec{w_1} = \begin{pmatrix}4\\4\\2\end{pmatrix}$

$$\vec{w_2} = \vec{v_2} - \text{proj}_{\vec{w_1}}(\vec{v_2}) = \begin{pmatrix}-4\\2\\4\end{pmatrix} - \frac{\begin{pmatrix}-4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}4\\4\\2\end{pmatrix}}{\begin{pmatrix}4\\4\\2\end{pmatrix}\cdot\begin{pmatrix}4\\4\\2\end{pmatrix}}\begin{pmatrix}4\\4\\2\end{pmatrix} = \begin{pmatrix}-4\\2\\4\end{pmatrix} - 0\begin{pmatrix}4\\4\\2\end{pmatrix} = \begin{pmatrix}-4\\2\\4\end{pmatrix}$$

(i.e. $\vec{v_2}$ and $\vec{v_1}$ were already orthogonal)

$$\vec{w_3} = \vec{v_3} - \text{proj}_{\vec{w_1}}(\vec{v_3}) - \text{proj}_{\vec{w_2}}(\vec{v_3}) = \begin{pmatrix}36\\0\\0\end{pmatrix} - \frac{\begin{pmatrix}36\\0\\0\end{pmatrix}\cdot\begin{pmatrix}4\\4\\2\end{pmatrix}}{\begin{pmatrix}4\\4\\2\end{pmatrix}\cdot\begin{pmatrix}4\\4\\2\end{pmatrix}}\begin{pmatrix}4\\4\\2\end{pmatrix} - \frac{\begin{pmatrix}36\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\4\end{pmatrix}}{\begin{pmatrix}-4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\4\end{pmatrix}}\begin{pmatrix}-4\\2\\4\end{pmatrix} = \begin{pmatrix}36\\0\\0\end{pmatrix} - \frac{144}{36}\begin{pmatrix}4\\4\\2\end{pmatrix} - \frac{(-144)}{36}\begin{pmatrix}-4\\2\\4\end{pmatrix}$$

$$= \begin{pmatrix}36\\0\\0\end{pmatrix} - 4\begin{pmatrix}4\\4\\2\end{pmatrix} + 4\begin{pmatrix}-4\\2\\4\end{pmatrix} = \begin{pmatrix}4\\-8\\8\end{pmatrix}$$

$$\vec{u_1} = \frac{1}{\sqrt{4^2+4^2+2^2}}\begin{pmatrix}4\\4\\2\end{pmatrix} = \frac{1}{\sqrt{36}}\begin{pmatrix}4\\4\\2\end{pmatrix} = \begin{pmatrix}4/6\\4/6\\2/6\end{pmatrix} = \boxed{\begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix}} = \vec{u_1}$$

$$\vec{u_2} = \frac{1}{\sqrt{(-4)^2+2^2+4^2}}\begin{pmatrix}-4\\2\\4\end{pmatrix} = \frac{1}{\sqrt{36}}\begin{pmatrix}-4\\2\\4\end{pmatrix} = \boxed{\begin{pmatrix}-2/3\\1/3\\2/3\end{pmatrix}} = \vec{u_2}$$

$$\vec{u_3} = \frac{1}{\sqrt{4^2+8^2+8^2}}\begin{pmatrix}4\\-8\\8\end{pmatrix} = \frac{1}{\sqrt{144}}\begin{pmatrix}4\\-8\\8\end{pmatrix} = \begin{pmatrix}4/12\\-8/12\\8/12\end{pmatrix} = \boxed{\begin{pmatrix}1/3\\-2/3\\2/3\end{pmatrix}} = \vec{u_3}$$

③ $A = \begin{pmatrix}4 & 5 & 0\\0 & 0 & -3\\3 & -5 & 0\end{pmatrix}$ $\quad \vec{v_1} = \begin{pmatrix}4\\0\\3\end{pmatrix}$ $\quad \vec{v_2} = \begin{pmatrix}5\\0\\-5\end{pmatrix}$ $\quad \vec{v_3} = \begin{pmatrix}0\\-3\\0\end{pmatrix}$

$\vec{w_1} = \begin{pmatrix}4\\0\\3\end{pmatrix}$

$$\vec{w_2} = \vec{v_3} - \text{proj}_{\vec{w_1}}(\vec{v_2}) = \begin{pmatrix}5\\0\\-5\end{pmatrix} - \frac{\begin{pmatrix}5\\0\\-5\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix}}{\begin{pmatrix}4\\0\\3\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix}}\begin{pmatrix}4\\0\\3\end{pmatrix} = \begin{pmatrix}5\\0\\-5\end{pmatrix} - \frac{5}{25}\begin{pmatrix}4\\0\\3\end{pmatrix} = \begin{pmatrix}5\\0\\-5\end{pmatrix} - \begin{pmatrix}4/5\\0\\3/5\end{pmatrix} = \begin{pmatrix}21/5\\0\\-28/5\end{pmatrix}$$

$$\vec{w_3} = \vec{v_3} - \text{proj}_{\vec{w_1}}(\vec{v_3}) - \text{proj}_{\vec{w_2}}(\vec{v_3}) = \begin{pmatrix}0\\-3\\0\end{pmatrix} - \frac{\begin{pmatrix}0\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix}}{\begin{pmatrix}4\\0\\3\end{pmatrix}\cdot\begin{pmatrix}4\\0\\3\end{pmatrix}}\begin{pmatrix}4\\0\\3\end{pmatrix} - \frac{\begin{pmatrix}0\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}21/5\\0\\-28/5\end{pmatrix}}{\begin{pmatrix}21/5\\0\\-28/5\end{pmatrix}\cdot\begin{pmatrix}21/5\\0\\-28/5\end{pmatrix}}\begin{pmatrix}21/5\\0\\-28/5\end{pmatrix} = \begin{pmatrix}0\\-3\\0\end{pmatrix} - 0\begin{pmatrix}4\\0\\3\end{pmatrix} - 0\begin{pmatrix}21/5\\0\\-28/5\end{pmatrix}$$

$$= \begin{pmatrix}0\\-3\\0\end{pmatrix}$$

③ continued

$$\vec{u_1} = \frac{1}{\sqrt{4^2+0^2+3^2}}\begin{pmatrix}4\\0\\3\end{pmatrix} = \frac{1}{\sqrt{25}}\begin{pmatrix}4\\0\\3\end{pmatrix} = \begin{pmatrix}4/5\\0\\3/5\end{pmatrix} \qquad \vec{v_1} = \begin{pmatrix}4\\0\\3\end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix}5\\0\\-5\end{pmatrix}, \quad \vec{v_3} = \begin{pmatrix}0\\-3\\0\end{pmatrix}$$

$$\vec{u_2} = \frac{1}{\sqrt{\left(\frac{21}{5}\right)^2+0^2+\left(\frac{28}{5}\right)^2}}\begin{pmatrix}21/5\\0\\-28/5\end{pmatrix} = \frac{1}{7}\begin{pmatrix}21/5\\0\\-28/5\end{pmatrix} = \begin{pmatrix}3/5\\0\\-4/5\end{pmatrix}$$

$$\vec{u_3} = \frac{1}{\sqrt{0^2+3^2+0^2}}\begin{pmatrix}0\\-3\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$R = \begin{pmatrix}\vec{u_1}\cdot\vec{v_1} & \vec{u_1}\cdot\vec{v_2} & \vec{u_1}\cdot\vec{v_3}\\ 0 & \vec{u_2}\cdot\vec{v_2} & \vec{u_2}\cdot\vec{v_3}\\ 0 & 0 & \vec{u_3}\cdot\vec{v_3}\end{pmatrix} = \begin{pmatrix}\frac{16}{5}+0+\frac{9}{5} & 4+0-3 & 0\\ 0 & 3+0+4 & 0\\ 0 & 0 & 3\end{pmatrix} = \begin{pmatrix}5&1&0\\0&7&0\\0&0&3\end{pmatrix}$$

QR factorization is therefore

$$\boxed{\begin{pmatrix}4&5&0\\0&0&-3\\3&-5&0\end{pmatrix} = \underbrace{\begin{pmatrix}4/5&3/5&0\\0&0&-1\\3/5&-4/5&0\end{pmatrix}}_{Q}\underbrace{\begin{pmatrix}5&1&0\\0&7&0\\0&0&3\end{pmatrix}}_{R}}$$

④ Find bases first.

$$\left(\begin{array}{ccc|c}2&3&0&0\\1&7&6&0\\-2&8&10&0\end{array}\right) \xrightarrow{R_1+R_3\to R_3} \left(\begin{array}{ccc|c}2&3&0&0\\1&7&6&0\\0&11&10&0\end{array}\right) \xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c}1&7&6&0\\2&3&0&0\\0&11&10&0\end{array}\right) \xrightarrow{-2R_1+R_2\to R_2} \left(\begin{array}{ccc|c}1&7&6&0\\0&-11&-12&0\\0&11&10&0\end{array}\right) \xrightarrow{R_2+R_3\to R_3}$$

$$\left(\begin{array}{ccc|c}1&7&6&0\\0&-11&-12&0\\0&0&-2&0\end{array}\right) \xrightarrow[-\frac{1}{2}R_3\to R_3]{-\frac{1}{11}R_2\to R_2} \left(\begin{array}{ccc|c}1&7&6&0\\0&1&\frac{12}{11}&0\\0&0&1&0\end{array}\right)$$

pivot in every column

$$\operatorname{im}T = \operatorname{span}\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\7\\8\end{pmatrix}, \begin{pmatrix}0\\6\\10\end{pmatrix}\right\}$$

$$\ker T = \{\vec{0}\}$$

Now find orthogonal basis of imT (i.e. Gram-Schmidt) — alternatively, you could have started w/ this step

$$\vec{v_1} = \begin{pmatrix}2\\1\\-2\end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix}3\\7\\8\end{pmatrix}, \quad \vec{v_3} = \begin{pmatrix}0\\6\\10\end{pmatrix}$$

$$\vec{w_1} = \begin{pmatrix}2\\1\\-2\end{pmatrix}$$

$$\vec{w_2} = \vec{v_2} - \operatorname{proj}_{\vec{w_1}}(\vec{v_2}) = \begin{pmatrix}3\\7\\8\end{pmatrix} - \frac{\begin{pmatrix}3\\7\\8\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}}{\begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}}\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}3\\7\\8\end{pmatrix} + \frac{+3}{9}\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}3\\7\\8\end{pmatrix} - \begin{pmatrix}2/3\\1/3\\-2/3\end{pmatrix} = \begin{pmatrix}7/3\\20/3\\26/3\end{pmatrix}$$

$$\vec{w_3} = \vec{v_3} - \operatorname{proj}_{\vec{w_1}}(\vec{v_3}) - \operatorname{proj}_{\vec{w_2}}(\vec{v_3}) = \begin{pmatrix}0\\6\\10\end{pmatrix} - \frac{\begin{pmatrix}0\\6\\10\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}}{\begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}}\begin{pmatrix}2\\1\\-2\end{pmatrix} - \frac{\begin{pmatrix}0\\6\\10\end{pmatrix}\cdot\begin{pmatrix}7/3\\20/3\\26/3\end{pmatrix}}{\begin{pmatrix}7/3\\20/3\\26/3\end{pmatrix}\cdot\begin{pmatrix}7/3\\20/3\\26/3\end{pmatrix}}\begin{pmatrix}7/3\\20/3\\26/3\end{pmatrix} = \begin{pmatrix}0\\6\\10\end{pmatrix} - \frac{-14}{9}\begin{pmatrix}2\\1\\-2\end{pmatrix} - \frac{\frac{380}{3}}{125}\begin{pmatrix}7/3\\20/3\\26/3\end{pmatrix}$$

$$= \begin{pmatrix}56/75\\4/5\\-142/75\end{pmatrix}$$

$$\text{orthogonal basis of } \operatorname{im}T = \left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}7/3\\20/3\\26/3\end{pmatrix}, \begin{pmatrix}56/75\\4/5\\-142/75\end{pmatrix}\right\}$$

⑤ Already looks close to QR factorization the right matrix is already upper triangular.
Do QR on first matrix.
Notice that the columns of the first matrix are orthogonal:

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = 1-1-1+1=0 \qquad \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = 1-1+1-1=0$$

$$\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = 1+1-1-1=0$$

We just need to normalize the vectors

$$\left\|\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2 \qquad \underset{\vec{v_1}}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}} \to \underset{\vec{u_1}}{\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}}$$

$$\left\|\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right\| = 2 \qquad \underset{\vec{v_2}}{\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}} \to \underset{\vec{u_2}}{\begin{pmatrix}1/2\\-1/2\\-1/2\\1/2\end{pmatrix}} \qquad \left\|\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\| = 2 \qquad \underset{\vec{v_3}}{\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}} \to \underset{\vec{u_3}}{\begin{pmatrix}1/2\\-1/2\\1/2\\-1/2\end{pmatrix}}$$

⑤ continued

R should be
$$\begin{pmatrix} \vec{u_1}\cdot\vec{v_1} & \vec{u_1}\cdot\vec{v_2} & \vec{u_1}\cdot\vec{v_3} \\ 0 & \vec{u_2}\cdot\vec{v_2} & \vec{u_2}\cdot\vec{v_3} \\ 0 & 0 & \vec{u_3}\cdot\vec{v_3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2} \\ 0 & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

So now we have
$$\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 6 \\ 0 & -8 & 10 \\ 0 & 0 & 12 \end{pmatrix}$$

i.e. this scales left matrix by factor of 2 to get the first matrix

Almost there → but diagonal entries of R need to be positive! Scale 2nd column of left matrix by −1, 2nd row of right matrix by −1

$$\Rightarrow \boxed{\underbrace{\begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{pmatrix}}_{Q}\underbrace{\begin{pmatrix} 2 & 4 & 6 \\ 0 & 8 & -10 \\ 0 & 0 & 12 \end{pmatrix}}_{R}}$$